

# Horoballs and iteration of holomorphic maps on bounded symmetric domains

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**ABSTRACT.** Given a fixed-point free compact holomorphic self-map  $f$  on a bounded symmetric domain  $D$ , which may be infinite dimensional, we establish the existence of a family  $\{H(\xi, \lambda)\}_{\lambda>0}$  of convex  $f$ -invariant domains at a point  $\xi$  in the boundary  $\partial D$  of  $D$ , which generalises completely Wolff's theorem for the open unit disc in  $\mathbb{C}$ . Further, we construct horoballs at  $\xi$  and show that they are exactly the  $f$ -invariant domains when  $D$  is of finite rank. Consequently, we show in the latter case that the limit functions of the iterates  $(f^n)$  with weakly closed range all accumulate in one single boundary component of  $\partial D$ .

*MSC:* 32H50; 32M15; 17C65; 58B12; 58C10

*Keywords:* Bounded symmetric domain, holomorphic map, horoball, iteration, Cartan domain

## 1. INTRODUCTION

The invariant domains and iteration of a holomorphic self-map on a one-dimensional bounded symmetric domain is well-understood. In particular, given a fixed-point free holomorphic self-map  $f$  on the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , the celebrated Wolff's theorem [29], which can be viewed as an analogue of the Schwarz lemma, states that there is a point  $\xi$  in the boundary  $\partial\mathbb{D}$  and a family  $\{H(\xi, \lambda)\}_{\lambda>0}$  of  $f$ -invariant domains, which covers  $\mathbb{D}$  and consists of Euclidean open discs in  $\mathbb{D}$  with closure internally tangent to  $\xi$ , in other words, they are horodiscs of horocentre  $\xi$ . An immediate consequence is the Denjoy-Wolff Theorem [11, 30] which asserts the convergence of the iterates  $(f^n)$  to the constant function  $h(\cdot) = \xi$ . We refer to [5] for a succinct exposition of the details and historical remarks.

Although both theorems have been extended completely to Euclidean balls [15] (see also [23]) and various forms of generalisation to other domains in higher dimension have been shown by several authors (e.g. [1, 4, 6, 9, 14, 18, 23]), a unified treatment for bounded symmetric domains of all dimensions and a description of the invariant domains resembling Wolff's horodiscs  $\{H(\xi, \lambda)\}_{\lambda>0}$  seem wanting, apart from some results in [8, 10, 24]. In this paper, we consider all bounded symmetric domains, including the infinite dimensional ones, and adopt an approach using Jordan theory to the question of invariant domains and iteration of holomorphic maps. This enables us to give a complete generalisation of Wolff's theorem to all bounded symmetric domains and a version of the Denjoy-Wolff Theorem for finite-rank domains, which also unifies and improves the results in [8, 10, 24].

Given a fixed-point free *compact* holomorphic self-map  $f$  on a bounded symmetric domain  $D$ , we establish in Theorem 2.4 the existence of convex  $f$ -invariant domains  $\{H(\xi, \lambda)\}_{\lambda>0}$  at some boundary point  $\xi$  of  $D$ . Further, we construct the *horoballs* at  $\xi$ , which generalise Wolff's horodiscs, and show in Theorem 5.12 and Remark 6.5 that in the finite-rank case (including finite dimensions),

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the invariant domains  $H(\xi, \lambda)$  are exactly the horoballs at  $\xi$  and affinely homeomorphic to  $D$ . Using this extension of Wolff's theorem to finite-rank bounded symmetric domains, we show in Theorem 6.3 that all limit functions of the iterates  $(f^n)$  with weakly closed range must accumulate in one single boundary component in the boundary  $\partial D$ . This generalises the Denjoy-Wolff Theorem for rank-one bounded symmetric domains, which are the Hilbert balls. For infinite dimensional domains, however, the result in Theorem 6.3 need not be true without the compactness condition on  $f$ , even for Hilbert balls [27]. All holomorphic self-maps  $f$  on finite dimensional bounded domains are compact. To achieve these results, we need various topological properties of finite-rank bounded symmetric domains, which are shown in Section 4 and Section 5, and may be of some independent interest. We conclude with Example 6.5 to show that, for a Möbius transformation  $g$  of a finite-rank domain  $D$ , a limit function of the iterates  $(g^n)$  can be a constant map or its image is a whole single boundary component of  $\partial D$ .

A special feature of the paper is the substantial use of the underlying Jordan structures of bounded symmetric domains and some detailed computation involving the Bergmann operators, Möbius transformations and Peirce projections. It is possible that such a Jordan approach may also be fruitful in tackling other problems in complex geometry including the infinite dimensional case.

We begin by explaining briefly the connections between bounded symmetric domains and Jordan theory, but refer to [7, 28] for more details. A bounded symmetric domain is a bounded open connected set  $D$  in a complex Banach space such that each point  $a \in D$  is an isolated fixed point of an involutive biholomorphic map  $s_a : D \rightarrow D$ , called a *symmetry* at  $a$ . The most important connection to Jordan theory for this work is Kaup's Riemann mapping theorem [19] asserting that every bounded symmetric domain is biholomorphic to the open unit ball of a JB\*-triple  $V$ , which is a complex Banach space equipped with a Jordan triple structure.

More precisely, a complex Banach space  $V$  is called a *JB\*-triple* if it admits a continuous triple product  $\{\cdot, \cdot, \cdot\} : V^3 \rightarrow V$  which is symmetric and linear in the outer variables, but conjugate linear in the middle variable, and satisfies

- (i) (Triple Identity)  $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\};$
- (ii)  $\|\exp it(a \square a)\| = 1$  for all  $t \in \mathbb{R}$ ;
- (iii)  $a \square a$  has non-negative spectrum;
- (iv)  $\|a \square a\| = \|a\|^2$

for  $a, b, c, x, y \in V$ , where the box operator  $a \square b : V \rightarrow V$  is defined by  $a \square b(\cdot) = \{a, b, \cdot\}$ .

Open unit balls of JB\*-triples are bounded symmetric domains and in the case of the complex unit disc  $\mathbb{D} \subset \mathbb{C}$ , the triple product in  $\mathbb{C}$  is given by  $\{a, b, c\} = a\bar{b}c$ , where  $\bar{b}$  is the complex conjugate of  $b$ . In fact, a Hilbert space  $V$  is a JB\*-triple, with triple product  $\{a, b, c\} = (\langle a, b \rangle c + \langle c, b \rangle a)/2$ , where  $\langle \cdot, \cdot \rangle$  is the inner product. More generally, the Banach space  $\mathcal{L}(H, K)$  of bounded linear operators between Hilbert spaces  $H$  and  $K$ , as well as C\*-algebras  $\mathcal{A} \subset \mathcal{L}(H, H)$ , are JB\*-triples with triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \quad (a, b, c \in \mathcal{L}(H, K) \text{ or } \mathcal{A})$$

where  $b^*$  denotes the adjoint operator of  $b$ .

Throughout the paper,  $D$  will always denote a bounded symmetric domain realised as the open unit ball of a JB\*-triple  $V$ , with boundary  $\partial D = \overline{D} \setminus D = \{v \in V : \|v\| = 1\}$ . Besides the box operator  $a \square b$  defined above, which satisfies  $\|a \square b\| \leq \|a\|\|b\|$ , a fundamental operator on a

JB\*-triple  $V$  is the *Bergmann operator*  $B(a, b) : V \rightarrow V$  defined by

$$B(a, b)(x) = x - 2\{a, b, x\} + \{a, \{b, x, b\}, a\} \quad (a, b, x \in V)$$

which is invertible if  $\|a \square b\| < 1$ . We note that

$$\|B(a, b)(x)\| \leq \|x\| + 2\|a\|\|b\|\|x\| + \|a\|^2\|b\|^2\|x\| = (1 + \|a\|\|b\|)^2\|x\|.$$

For  $a \in D$ , the operator  $B(a, a)$  has non-negative spectrum and hence the square roots  $B(a, a)^{\pm 1/2}$  exist and moreover, we have the useful identity

$$\|B(a, a)^{-1/2}\| = \frac{1}{1 - \|a\|^2} \quad (1.1)$$

(cf. [7, Proposition 3.2.13]). The Bergmann operator  $B(a, b)$  on  $\mathcal{L}(H, H)$  can be written as

$$B(a, b)(x) = (\mathbf{1} - ab^*)x(\mathbf{1} - b^*a) \quad (1.2)$$

where  $\mathbf{1}$  denotes the identity operator on  $H$  [7, p. 191]. We note that  $\|\mathbf{1} - a^*a\| \leq 1$  for  $\|a\| \leq 1$ . On the complex plane  $\mathbb{C}$ , we have  $B(a, b)(x) = (1 - a\bar{b})^2x$ . Wolff's horodisc  $H(\xi, \lambda)$ , which is a one-dimensional horoball with horocentre  $\xi$ , has the form

$$H(\xi, \lambda) = t_\lambda^2\xi + (1 - t_\lambda^2)\mathbb{D}, \quad t_\lambda^2 = \lambda/(1 + \lambda)$$

and in terms of the Bergmann operator, it takes the form

$$H(\xi, \lambda) = t_\lambda^2\xi + B(t_\lambda\xi, t_\lambda\xi)^{1/2}\mathbb{D}.$$

It is the latter form of the horodisc  $H(\xi, \lambda)$  which will be generalised to all bounded symmetric domains of finite rank.

## 2. INVARIANT DOMAINS IN BOUNDED SYMMETRIC DOMAINS

To explain further the underlying idea in our construction of invariant domains and horoballs, we begin with a fixed-point free holomorphic self-map  $f$  on the unit disc  $\mathbb{D}$  in the complex plane. The key in Wolff's theorem is to produce a sequence  $(z_k)$  in  $\mathbb{D}$  converging to a boundary point  $\xi$ , which is used to construct the  $f$ -invariant domains in the following way. For each  $\lambda > 0$ , let  $D_k(\lambda)$  be a Poincaré disc (for sufficiently large  $k$ ), which is the open disc centred at  $z_k$ , with radius  $\tanh^{-1} r_k$ , measured by the Poincaré distance  $\kappa$ :

$$D_k(\lambda) = \{z \in \mathbb{D} : \kappa(z, z_k) < \tanh^{-1} r_k\}$$

where  $r_k \in (0, 1)$  satisfies  $1 - r_k^2 = \lambda(1 - |z_k|^2)$ . The Poincaré disc is the image  $g_{z_k}(\mathbb{D}(0, r_k))$  of the Euclidean disc  $\mathbb{D}(0, r_k) = \{z \in \mathbb{D} : |z| < r_k\}$ , under the Möbius transformation

$$g_{z_k}(z) = \frac{z + z_k}{1 + z\bar{z}_k} \quad (z \in \mathbb{D}).$$

The  $f$ -invariant domain  $H(\xi, \lambda)$  is then (the interior of) the '*limit*' of the Poincaré discs  $D_k(\lambda)$  and is given by

$$H(\xi, \lambda) = \left\{ z \in \mathbb{D} : \frac{|1 - z\bar{\xi}|^2}{1 - |z|^2} < \frac{1}{\lambda} \right\} = \frac{\lambda}{1 + \lambda}\xi + \frac{1}{1 + \lambda}\mathbb{D} \quad (2.1)$$

which is a disc centred at  $\frac{\lambda}{1 + \lambda}\xi$  with radius  $\frac{1}{1 + \lambda}$  and as noted earlier, its boundary is a horocycle with horocentre  $\xi$ . A crucial observation is that this invariant domain is identical with the sets

$$H(\xi, \lambda) = \left\{ z \in \mathbb{D} : \lim_k \frac{|1 - z\bar{z}_k|^2}{1 - |z|^2} < \frac{1}{\lambda} \right\} = \left\{ z \in \mathbb{D} : \lim_k \frac{1 - |z_k|^2}{1 - |g_{-z_k}(z)|^2} < \frac{1}{\lambda} \right\}. \quad (2.2)$$

Given a bounded symmetric domain realised as the open unit ball  $D$  of a JB\*-triple  $V$ , the Möbius transformations on  $D$  and the Kobayashi distance, which generalises the Poincaré distance in  $\mathbb{D}$ , have an explicit Jordan description and the observation in (2.2) enables us to extend Wolff's theorem completely to  $D$ . We carry this out in this section.

We first need some Jordan tools. Let  $D$  be the open unit ball of a JB\*-triple  $V$  and let  $a \in D$ . The Möbius transformation  $g_a : D \rightarrow D$ , induced by  $a$ , is a biholomorphic map given by

$$g_a(z) = a + B(a, a)^{1/2}(\mathbf{1} + z \square a)^{-1}(z) \quad (z \in D)$$

with inverse  $g_{-a}$ , where  $\mathbf{1}$  denotes the identity operator on  $V$ . The Kobayashi distance  $\kappa(x, y)$  between two points  $x$  and  $y$  in  $D$  can be described in terms of a Möbius transformation:

$$\kappa(x, y) = \tanh^{-1} \|g_{-x}(y)\| = \tanh^{-1} \|g_{-y}(x)\|.$$

By the Schwarz lemma, a holomorphic self-map  $f$  on  $D$  satisfies

$$\|g_{-f(x)}(f(y))\| \leq \|g_{-x}(y)\| \quad (2.3)$$

that is,  $f$  is  $\kappa$ -nonexpansive. We will often use the following norm estimate

$$\frac{1}{1 - \|g_{-z}(a)\|^2} = \|B(a, a)^{-1/2}B(a, z)B(z, z)^{-1/2}\| \quad (a, z \in D) \quad (2.4)$$

which has been proved in [24] (see also [7, Lemma 3.2.17]).

The aforementioned sequence  $(z_k)$  in Wolff's theorem has a limit point  $\xi$  by relative compactness of  $\mathbb{D}$ . An infinite dimensional domain  $D$  need not be relatively compact and the existence of limit points is not guaranteed. For this reason, we consider *compact* maps  $f : D \rightarrow D \subset V$ , which are the ones having relatively compact image  $f(D)$ , that is, the closure  $\overline{f(D)}$  is compact in  $V$ . All continuous self-maps on a finite dimensional bounded domain are necessarily compact.

Now let  $f$  be a compact fixed-point free holomorphic self-map on  $D$ . Choose an increasing sequence  $(\alpha_k)$  in  $(0, 1)$  with limit 1. Then  $\alpha_k f$  maps  $D$  strictly inside itself and by the fixed-point theorem of Earle and Hamilton [12], we have  $\alpha_k f(z_k) = z_k$  for some  $z_k \in D$ . Note that  $z_k \neq 0$ . Since  $f(D)$  is relatively compact, we may assume, by choosing a subsequence if necessary, that  $(z_k)$  converges to a point  $\xi \in \overline{D}$ . Since  $f$  has no fixed point in  $D$ , the point  $\xi$  must lie in the boundary  $\partial D$ .

Generalizing Wolff's one-dimensional horodisc, we now define a *horoball at  $\xi$*  as (the interior of) a limit of Kobayashi balls as follows. Alternative descriptions of horospheres in various domains have been given in [1, 2, 25].

Given  $\lambda > 0$ , pick a sequence  $(r_k)$  in  $(0, 1)$  such that

$$1 - r_k^2 = \lambda(1 - \|z_k\|^2)$$

from some  $k$  onwards. For each  $r_k$ , define a Kobayashi ball, centred at  $z_k$ , by

$$D_k(\lambda) = \{z \in D : \kappa(z, z_k) < \tanh^{-1} r_k\} = \{x \in D : \|g_{-z_k}(x)\| < r_k\} = g_{z_k}(D(0, r_k)) \quad (2.5)$$

where  $D(0, r_k) = \{z \in V : \|z\| < r_k\}$ . The Kobayashi balls generalise the one-dimensional Poincaré discs.

For  $x \in \overline{D}$  and  $0 < r < 1$ , using (2.5) and the following formula

$$g_z(rx) = (1 - r^2)B(rz, rz)^{-1/2}(z) + rB(z, z)^{1/2}B(rz, rz)^{-1/2}g_{rz}(x), \quad (2.6)$$

as in [24, Proposition 2.3], one can write

$$D_k(\lambda) = (1 - r_k^2)B(r_k z_k, r_k z_k)^{-1/2}(z_k) + r_k B(z_k, z_k)^{1/2}B(r_k z_k, r_k z_k)^{-1/2}(D). \quad (2.7)$$

In particular,  $D_k(\lambda)$  is a convex domain.

Now we define the *limit* of the sequence  $(D_k(\lambda))_k$  of Kobayashi balls as the following set:

$$S(\xi, \lambda) = \{x \in \overline{D} : x = \lim_k x_k, x_k \in D_k(\lambda)\} \quad (2.8)$$

which contains  $\xi$  since  $z_k \in D_k(\lambda)$ , and is also convex because each  $D_k(\lambda)$  is. We call  $S(\xi, \lambda)$  a *closed horoball ball* at  $\xi$ , its interior  $S_0(\xi, \lambda)$  a *horoball* at  $\xi$  and will show that, for finite rank  $D$ , the latter resembles Wolff's horodisc and indeed,  $S_0(\xi, \lambda) = H(\xi, \lambda)$  in (2.1) for  $D = \mathbb{D}$ . Since  $D$  is the interior of  $\overline{D}$ , we have  $S_0(\xi, \lambda) \subset D$ .

We first construct invariant domains using the observation in (2.2).

**Lemma 2.1.** *The function  $F : D \rightarrow [0, \infty)$  given by*

$$F(x) = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} \quad (x \in D)$$

*is well-defined and continuous.*

*Proof.* For each  $x \in D$ , we have

$$\begin{aligned} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} &= \|B(x, x)^{-1/2} B(x, z_k) B(z_k, z_k)^{-1/2} (1 - \|z_k\|^2)\| \\ &\leq \|B(x, x)^{-1/2} B(x, z_k)\| \|B(z_k, z_k)^{-1/2} (1 - \|z_k\|^2)\| \\ &= \|B(x, x)^{-1/2} B(x, z_k)\| \leq \frac{(1 + \|x\| \|z_k\|)^2}{1 - \|x\|^2} \leq \frac{1 + \|x\|}{1 - \|x\|} \end{aligned}$$

and hence the defining sequence for  $F(x)$  is bounded. Therefore  $F$  is well-defined.

For continuity, let  $x, y \in D$  and write  $x_k = \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2}$ , also  $y_k = \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(y)\|^2}$ . Then we have

$$|x_k - y_k| \leq \|B(x, x)^{-1/2} B(x, z_k) - B(y, y)^{-1/2} B(y, z_k)\|$$

which gives

$$\begin{aligned} |F(x) - F(y)| &= |\limsup_k x_k - \limsup_k y_k| \\ &\leq \limsup_k \|B(x, x)^{-1/2} B(x, z_k) - B(y, y)^{-1/2} B(y, z_k)\| \\ &= \|B(x, x)^{-1/2} B(x, \xi) - B(y, y)^{-1/2} B(y, \xi)\| \end{aligned}$$

since  $(z_k)$  norm converges to  $\xi$ . Now continuity of  $F$  follows from that of the function  $h(x) = \|B(x, x)^{-1/2} B(x, \xi)\|$  on  $D$ . □

*Remark 2.2.* We will show in Corollary 5.15 that  $F^{-1}[0, r) \neq \emptyset$  for each  $r > 0$ .

A similar computation as in the previous proof yields the following result.

**Lemma 2.3.** *Let  $(x_k)$  be a sequence in  $D$  norm converging to  $x \in D$ . Then we have*

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x_k)\|^2} = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2}.$$

The following result is a first generalisation of Wolff's theorem to all bounded symmetric domains.

**Theorem 2.4.** *Let  $f$  be a fixed-point free compact holomorphic self-map on a bounded symmetric domain  $D$ . Then there is a sequence  $(z_k)$  in  $D$  converging to a boundary point  $\xi \in \overline{D}$  such that, for each  $\lambda > 0$ , the set*

$$H(\xi, \lambda) = \left\{ x \in D : \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} < \frac{1}{\lambda} \right\}$$

*is a non-empty convex domain and  $f$ -invariant, that is,  $f(H(\xi, \lambda)) \subset H(\xi, \lambda)$ . Moreover,  $D = \bigcup_{\lambda > 0} H(\xi, \lambda)$  and  $0 \in \bigcap_{\lambda < 1} H(\xi, \lambda)$ .*

*Proof.* By Lemma 2.1, the set  $H(\xi, \lambda)$  is open and by Remark 2.2, it is non-empty. To see that  $H(\xi, \lambda)$  is convex, let  $x, y \in H(\xi, \lambda)$  and  $0 < \alpha < 1$ . We show  $\alpha x + (1 - \alpha)y \in H(\xi, \lambda)$ . There exists  $k_0$  such that  $k \geq k_0$  implies

$$\frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2}, \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(y)\|^2} < \frac{1}{\beta} < \frac{1}{\lambda}$$

for some  $\beta$  satisfying  $\beta(1 - \|z_k\|^2) = 1 - s_k^2$  with  $s_k \in (0, 1)$ . This gives  $\|g_{-z_k}(x)\| < s_k$  and  $\|g_{-z_k}(y)\| < s_k$ , that is,  $x$  and  $y$  belong to the Kobayashi ball  $D_k(\beta)$  as defined in (2.5). Since  $D_k(\beta)$  is convex, the element  $w = \alpha x + (1 - \alpha)y$  is in  $D_k(\beta)$  for  $k \geq k_0$ . Therefore

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(w)\|^2} \leq \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - s_k^2} = \frac{1}{\beta} < \frac{1}{\lambda}$$

and  $w \in H(\xi, \lambda)$ .

For  $f$ -invariance, let  $x \in H(\xi, \lambda)$ . We need to show  $f(x) \in H(\xi, \lambda)$ . Let  $f_k = \alpha_k f$  be as before. Then by (2.3), we have

$$\|g_{-z_k}(f_k(x))\| = \|g_{-f_k(z_k)}(f_k(x))\| \leq \|g_{-z_k}(x)\|$$

and

$$\frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(f_k(x))\|^2} \leq \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2}.$$

Hence Lemma 2.3 implies

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(f(x))\|^2} = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(f_k(x))\|^2} \leq \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} < \frac{1}{\lambda}$$

which gives  $f(x) \in H(\xi, \lambda)$ .

Finally, for each  $y \in D$ , we have  $y \in H(\xi, \lambda)$  whenever  $F(y) < 1/\lambda$ . Since  $F(0) = 1$ , we have  $0 \in H(\xi, \lambda)$  for  $\lambda < 1$ . This proves the last assertion.  $\square$

We will show that the domain  $H(\xi, \lambda)$  resembles Wolff's horodisc for finite rank  $D$ , which is a second generalisation of Wolff's theorem. This is achieved by showing that  $H(\xi, \lambda)$  is identical with the horoball  $S_0(\xi, \lambda)$  and by giving an explicit description of  $S_0(\xi, \lambda)$ . We first point out that the result in [8, Lemma 4.2] (see also [8, Theorem 4.6]) for Lie balls is valid for all bounded symmetric domains, which shows the invariance of the larger convex set  $S(\xi, \lambda) \cap D$ . This is stated below.

**Theorem 2.5.** *Let  $f$  be a fixed-point free compact holomorphic self-map on a bounded symmetric domain. Then there is a sequence  $(z_k)$  in  $D$  converging to a boundary point  $\xi \in \overline{D}$  such that, for each  $\lambda > 0$ , we have  $f(S(\xi, \lambda) \cap D) \subset S(\xi, \lambda) \cap D$ .*

The following lemma shows the inclusion  $H(\xi, \lambda) \subset S(\xi, \lambda) \cap D$ .

**Lemma 2.6.**  $H(\xi, \lambda) \subset S_0(\xi, \lambda)$  and  $S(\xi, \lambda) \cap D \subset \{x \in D : F(x) \leq 1/\lambda\}$ .

*Proof.* Let  $x \in H(\xi, \lambda)$ . Then we have, from some  $k$  onwards,

$$\frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} < \frac{1}{\lambda} = \frac{1 - \|z_k\|^2}{1 - r_k^2}$$

which implies  $\|g_{-z_k}(x)\| < r_k$ , that is,  $x \in D_k(\lambda)$ . Hence  $x \in S(\xi, \lambda)$ . We have shown  $H(\xi, \lambda) \subset S(\xi, \lambda)$  and therefore  $H(\xi, \lambda) \subset S_0(\xi, \lambda)$  since  $H(\xi, \lambda)$  is open.

For the second assertion, let  $x \in S(\xi, \lambda) \cap D$  with  $x = \lim_k x_k$  and  $x_k \in D_k(\lambda)$ . Since  $\|g_{-z_k}(x_k)\| < r_k$  for each  $k$ , we have from Lemma 2.3 that

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x_k)\|^2} \leq \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - r_k^2} = \frac{1}{\lambda}.$$

This proves  $S(\xi, \lambda) \cap D \subset \{x : F(x) \leq 1/\lambda\}$ .  $\square$

**Example 2.7.** Let  $D$  be the open unit ball of the JB\*-triple  $C(\Omega)$  of complex continuous functions on a compact Hausdorff space  $\Omega$ , with Jordan triple product  $\{x, y, z\} = x\bar{y}z$  where  $\bar{y}$  denotes the complex conjugate of the function  $y \in C(\Omega)$ . For  $a, b \in D$ , the Bergmann operator  $B(a, b)$  is given by a product of functions:

$$B(a, b)(z) = (\mathbf{1} - a\bar{b})^2 z \quad (z \in C(\Omega))$$

where  $\mathbf{1}$  denotes the constant function with value 1 (cf. [7, Example 3.2.12]) and we have

$$\|B(a, b)\| = \|(\mathbf{1} - a\bar{b})^2\| = \sup\{|1 - a(\omega)\bar{b}(\omega)|^2 : \omega \in \Omega\}.$$

Let  $(z_k)$  be a sequence in  $D$  converging to  $\xi \in \partial D$  as before. Then

$$\|B(x, x)^{-1/2} B(x, z_k) B(z_k, z_k)^{-1/2}\| (1 - \|z_k\|^2) = \left\| \frac{(\mathbf{1} - x\bar{z}_k)^2 (1 - \|z_k\|^2)}{(\mathbf{1} - |x|^2)(1 - |z_k|^2)} \right\|$$

Since  $\left\| \frac{1 - \|z_k\|^2}{1 - |z_k|^2} \right\| \leq 1$  and the sequence  $(\mathbf{1} - x\bar{z}_k)$  converges to  $\mathbf{1} - x\bar{\xi}$  in  $C(\Omega)$ , we have

$$\limsup_k \|B(x, x)^{-1/2} B(x, z_k) B(z_k, z_k)^{-1/2}\| (1 - \|z_k\|^2) = \limsup_k \left\| \frac{(\mathbf{1} - x\bar{\xi})^2}{1 - |x|^2} \frac{1 - \|z_k\|^2}{1 - |z_k|^2} \right\|$$

and for  $\lambda > 0$ ,

$$H(\xi, \lambda) = \left\{ x \in D : \limsup_k \left\| \frac{(\mathbf{1} - x\bar{\xi})^2}{1 - |x|^2} \left( \frac{1 - \|z_k\|^2}{1 - |z_k|^2} \right) \right\| < \frac{1}{\lambda} \right\}$$

which reduces to the horodisc in (2.1) if  $\Omega$  is a singleton in which case the function  $\frac{1 - \|z_k\|^2}{1 - |z_k|^2} = \mathbf{1}$ .

### 3. PEIRCE DECOMPOSITIONS

To study the invariant domains  $H(\xi, \lambda)$  and the horoballs  $S_0(\xi, \lambda)$  in depth, we need to develop more tools from the ambient Jordan structures. We begin with the Peirce decompositions of JB\*-triples. An element  $e$  in a JB\*-triple  $V$  is called a *tripotent* if  $\{e, e, e\} = e$ . For a nonzero tripotent  $e$ , we have  $\|e\| = 1$ . A nonzero tripotent  $e$  is called *minimal* if  $\{e, V, e\} = \mathbb{C}e$ . Two elements  $a, b \in V$  are said to be mutually (triple) *orthogonal* if  $a \square b = b \square a = 0$ , which is equivalent to  $a \square b = 0$  [7, Lemma 1.2.32]. In this case, we have  $\|a + b\| = \max\{\|a\|, \|b\|\}$  [7, Corollary 3.1.21]. Evidently, if  $e$  and  $c$  are mutually orthogonal tripotents, then  $e + c$  is also a tripotent.

A tripotent  $e \in V$  induces a *Peirce decomposition*

$$V = V_0(e) \oplus V_1(e) \oplus V_2(e)$$

where each  $V_k(e)$ , called the *Peirce  $k$ -space*, is an eigenspace

$$V_k(e) = \left\{ z \in V : (e \square e)(z) = \frac{k}{2} z \right\} \quad (k = 0, 1, 2)$$

of the operator  $e \square e$ , and is the range of the contractive projection  $P_k(e) : V \longrightarrow V$  given by

$$P_0(e) = B(e, e); \quad P_1(e) = 4(e \square e - (e \square e)^2); \quad P_2(e) = 2(e \square e)^2 - e \square e.$$

We call  $P_k(e)$  the *Peirce  $k$ -projection* and refer to [7, p. 32] for more detail.

By [7, Corollary 1.2.46], a tripotent  $c$  is orthogonal to  $e$  if and only if  $c \in V_0(e)$ . A tripotent  $e$  is called *maximal* if  $V_0(e) = \{0\}$ .

Let  $\{e_1, \dots, e_n\}$  be a family of mutually orthogonal tripotents in a JB\*-triple  $V$ . For  $i, j \in \{0, 1, \dots, n\}$ , the *joint Peirce space*  $V_{ij}$  is defined by

$$V_{ij} := V_{ij}(e_1, \dots, e_n) = \{z \in V : 2\{e_k, e_k, z\} = (\delta_{ik} + \delta_{jk})z \text{ for } k = 1, \dots, n\},$$

where  $\delta_{ij}$  is the Kronecker delta and  $V_{ij} = V_{ji}$ .

The decomposition

$$V = \bigoplus_{0 \leq i \leq j \leq n} V_{ij}$$

is called a *joint Peirce decomposition* (cf. [22]). More verbosely,

$$\begin{aligned} V_{ii} &= V_2(e_i), & i &= 1, \dots, n; \\ V_{ij} &= V_{ji} = V_1(e_i) \cap V_1(e_j), & 1 \leq i < j \leq n; \\ V_{i0} &= V_{0i} = V_1(e_i) \cap \bigcap_{j \neq i} V_0(e_j), & i &= 1, \dots, n; \\ V_{00} &= V_0(e_1) \cap \dots \cap V_0(e_n). \end{aligned}$$

The Peirce multiplication rules

$$\{V_{ij}, V_{jk}, V_{kl}\} \subset V_{il} \quad \text{and} \quad V_{ij} \square V_{pq} = \{0\} \quad \text{for} \quad i, j \notin \{p, q\}$$

hold, where we define  $\{A, B, C\} = \{\{a, b, c\} : a \in A, b \in B, c \in C\}$  and  $A \square B = \{a \square b : a \in A, b \in B\}$  for  $A, B, C \subset V$ . The contractive projection  $P_{ij}(e_1, \dots, e_n)$  from  $V$  onto  $V_{ij}(e_1, \dots, e_n)$  is called a *joint Peirce projection* which satisfies

$$P_{ij}(e_1, \dots, e_n)(e_k) = \begin{cases} 0 & (i \neq j) \\ \delta_{ik} e_k & (i = j). \end{cases} \quad (3.1)$$

We shall simplify the notation  $P_{ij}(e_1, \dots, e_n)$  to  $P_{ij}$  if the tripotents  $e_1, \dots, e_n$  are understood. For a single tripotent  $e \in V$ , we have  $P_{11}(e) = P_2(e)$ ,  $P_{10}(e) = P_1(e)$  and  $P_{00}(e) = P_0(e)$ .

Let  $M = \{0, 1, \dots, n\}$  and  $N \subset \{1, \dots, n\}$ . The Peirce  $k$ -spaces of the tripotent  $e_N = \sum_{i \in N} e_i$  are given by

$$V_2(e_N) = \bigoplus_{i, j \in N} V_{ij}, \quad (3.2)$$

$$V_1(e_N) = \bigoplus_{\substack{i \in N \\ j \in M \setminus N}} V_{ij}, \quad (3.3)$$

$$V_0(e_N) = \bigoplus_{i, j \in M \setminus N} V_{ij}. \quad (3.4)$$



**Lemma 3.1.** *Let  $e_1, \dots, e_r$  be mutually orthogonal tripotents in a  $JB^*$ -triple  $V$  and let  $J \subset \{1, \dots, r\}$  be non-empty. Then we have*

- (i)  $P_{ij}(e_s : s \in J) = P_{ij}(e_1, \dots, e_r)$  for  $i, j \in J$ ,
- (ii)  $P_{0j}(e_s : s \in J) = \sum_{i \in \{0, 1, \dots, r\} \setminus J} P_{ij}(e_1, \dots, e_r)$  for  $j \in J$ ,
- (iii)  $P_{00}(e_s : s \in J) = \sum_{\substack{i \leq j \\ i, j \in \{0, 1, \dots, r\} \setminus J}} P_{ij}(e_1, \dots, e_r).$

*Proof.* By re-ordering the indices, we may assume  $J = \{1, \dots, m-1\}$  for some  $m \in \{2, \dots, r\}$  and it amounts to proving

- (i)  $P_{ij}(e_1, \dots, e_{m-1}) = P_{ij}(e_1, \dots, e_r)$  for  $1 \leq i \leq j \leq m-1$ ,
- (ii)  $P_{0j}(e_1, \dots, e_{m-1}) = \sum_{i \in \{0\} \cup M} P_{ij}(e_1, \dots, e_r)$  for  $1 \leq j \leq m-1$ ,
- (iii)  $P_{00}(e_1, \dots, e_{m-1}) = \sum_{\substack{i \leq j \\ i, j \in \{0\} \cup M}} P_{ij}(e_1, \dots, e_r),$

where  $M = \{m, m+1, \dots, r\}$ .

For  $1 \leq i < j \leq m-1$  in (i), we have

$$P_{ij}(e_1, \dots, e_{m-1}) = P_1(e_i)P_1(e_j) = P_{ij}(e_1, \dots, e_r).$$

For  $i = j$ , we have  $P_{ij}(e_1, \dots, e_{m-1}) = P_2(e_i)P_2(e_j) = P_{ij}(e_1, \dots, e_r).$

To show (ii), we use the two families  $\{e_1, \dots, e_{m-1}\}$  and  $\{e_1, \dots, e_r\}$  to decompose  $V$ . In terms of projections this gives

$$\sum_{0 \leq i \leq k \leq m-1} P_{ik}(e_1, \dots, e_{m-1}) = \sum_{0 \leq i \leq k \leq r} P_{ik}(e_1, \dots, e_r). \quad (3.5)$$

Fix  $j \in \{1, \dots, m-1\}$ . Applying the Peirce 1-projection with respect to the tripotent  $e_j$  on both sides of (3.5) and using (i) gives

$$\begin{aligned} & P_{0j}(e_1, \dots, e_{m-1}) + \sum_{k \in \{1, \dots, m-1\} \setminus \{j\}} P_{jk}(e_1, \dots, e_{m-1}) \\ &= P_{0j}(e_1, \dots, e_r) + \sum_{k \in \{1, \dots, r\} \setminus \{j\}} P_{jk}(e_1, \dots, e_r) \\ &= P_{0j}(e_1, \dots, e_r) + \sum_{k \in \{1, \dots, m-1\} \setminus \{j\}} P_{jk}(e_1, \dots, e_{m-1}) + \sum_{k \in \{m, m+1, \dots, r\}} P_{jk}(e_1, \dots, e_r). \end{aligned}$$

Hence

$$P_{0j}(e_1, \dots, e_{m-1}) = P_{0j}(e_1, \dots, e_r) + \sum_{k \in \{m, m+1, \dots, r\} = M} P_{jk}(e_1, \dots, e_r) = \sum_{i \in \{0\} \cup M} P_{ij}(e_1, \dots, e_r).$$

To see (iii), let  $1 \leq p < q \leq r$ . From the definition of the joint Peirce spaces, we have  $V_{pq} \subset V_0(e_s)$  for every  $s \in \{1, \dots, r\} \setminus \{p, q\}$ . Therefore

$$\sum_{m \leq i < k \leq r} P_1(e_i)P_1(e_k) = P_0(e_1) \dots P_0(e_{m-1}) \sum_{m \leq i < k \leq r} P_1(e_i)P_1(e_k). \quad (3.6)$$

Applying  $P_{00}(e_1, \dots, e_{m-1})$  to both sides of (3.5) and using (3.6) gives

$$\begin{aligned} P_{00}(e_1, \dots, e_{m-1}) &= P_{00}(e_1, \dots, e_{m-1}) \sum_{0 \leq i \leq k \leq r} P_{ik}(e_1, \dots, e_r) \\ &= P_{00}(e_1, \dots, e_{m-1}) \left( P_{00}(e_1, \dots, e_r) + \sum_{k=1}^r P_{0k}(e_1, \dots, e_r) + \sum_{0 < i < k \leq r} P_{ik}(e_1, \dots, e_r) + \sum_{i=1}^r P_{ii}(e_1, \dots, e_r) \right) \\ &= P_0(e_1) \dots P_0(e_{m-1}) \left( P_0(e_1) \dots P_0(e_r) + \sum_{k=1}^r P_1(e_k) \prod_{\substack{i \neq k \\ 1 \leq i \leq r}} P_0(e_i) + \sum_{0 < i < k \leq r} P_1(e_i)P_1(e_k) + \sum_{i=1}^r P_2(e_i) \right) \\ &= P_0(e_1) \dots P_0(e_r) + \sum_{k=m}^r P_1(e_k) \prod_{\substack{i \neq k \\ 1 \leq i \leq r}} P_0(e_i) + \sum_{m \leq i < k \leq r} P_1(e_i)P_1(e_k) + \sum_{i=m}^r P_2(e_i) \\ &= P_{00}(e_1, \dots, e_r) + \sum_{k=m}^r P_{0k}(e_1, \dots, e_r) + \sum_{m \leq i < k \leq r} P_{ik}(e_1, \dots, e_r) + \sum_{i=m}^r P_{ii}(e_1, \dots, e_r) \\ &= \sum_{\substack{i \leq j \\ i, j \in \{0\} \cup M}} P_{ij}(e_1, \dots, e_r). \end{aligned}$$

□

The Peirce projections provide a very useful formulation of the Bergmann operators. Let  $e_1, \dots, e_n$  be mutually triple orthogonal tripotents in a JB\*-triple  $V$  and let  $x = \sum_{i=1}^n \lambda_i e_i$  with  $\lambda_i \in \mathbb{C}$ . Then the Bergmann operator  $B(x, x)$  satisfies

$$B(x, x) = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)(1 - |\lambda_j|^2) P_{ij}. \quad (3.7)$$

where we set  $\lambda_0 = 0$  and  $P_{ij} = P_{ij}(e_1, \dots, e_n)$ . This gives the following formulae for the square roots

$$B(x, x)^{1/2} = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)^{1/2} (1 - |\lambda_j|^2)^{1/2} P_{ij} \quad (\|x\| < 1) \quad (3.8)$$

$$B(x, x)^{-1/2} = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)^{-1/2} (1 - |\lambda_j|^2)^{-1/2} P_{ij} \quad (\|x\| < 1). \quad (3.9)$$

#### 4. FINITE-RANK BOUNDED SYMMETRIC DOMAINS

Finite-rank bounded symmetric domains are (biholomorphically equivalent to) open unit balls of finite-rank JB\*-triples. To describe them, we first recall the definition of the rank of a JB\*-triple.

A closed subspace  $E$  of a JB\*-triple  $V$  is called a *subtriple* if  $a, b, c \in E$  implies  $\{a, b, c\} \in E$ . The Peirce spaces  $V_{ij}$  defined before are subtriples of  $V$ . For each  $a \in V$ , let  $V(a)$  be the smallest closed subtriple of  $V$  containing  $a$ . For  $V \neq \{0\}$ , the *rank* of  $V$  is defined to be

$$r(V) = \sup\{\dim V(a) : a \in V\} \in \mathbb{N} \cup \{\infty\}.$$

A (nonzero) JB\*-triple  $V$  has *finite rank*, that is,  $r(V) < \infty$  if, and only if,  $V$  is a reflexive Banach space [20, Proposition 3.2]. In this case, its rank  $r(V)$  is the (unique) cardinality of a maximal family of mutually orthogonal minimal tripotents and  $V$  is an  $\ell^\infty$ -sum of a finite number of finite-rank Cartan factors, which can be infinite dimensional. There are six types of finite-rank Cartan factors, listed below.

- Type I  $\mathcal{L}(\mathbb{C}^r, H)$  ( $r = 1, 2, \dots$ ),
- Type II  $\{z \in \mathcal{L}(\mathbb{C}^r, \mathbb{C}^r) : z^t = -z\}$  ( $r = 5, 6, \dots$ ),
- Type III  $\{z \in \mathcal{L}(\mathbb{C}^r, \mathbb{C}^r) : z^t = z\}$  ( $r = 2, 3, \dots$ ),
- Type IV spin factor,
- Type V  $M_{1,2}(\mathcal{O}) = 1 \times 2$  matrices over the Cayley algebra  $\mathcal{O}$ ,
- Type VI  $M_3(\mathcal{O}) = 3 \times 3$  hermitian matrices over  $\mathcal{O}$ ,

where  $\mathcal{L}(\mathbb{C}^r, H)$  is the JB\*-triple of linear operators from  $\mathbb{C}^r$  to a Hilbert space  $H$  and  $z^t$  denotes the transpose of  $z$  in the JB\*-triple  $\mathcal{L}(\mathbb{C}^r, \mathbb{C}^r)$  of  $r \times r$  complex matrices. A *spin factor* is a JB\*-triple  $V$  equipped with a complete inner product  $\langle \cdot, \cdot \rangle$  and a conjugation  $*$  :  $V \rightarrow V$  satisfying

$$\langle x^*, y^* \rangle = \langle y, x \rangle \quad \text{and} \quad \{x, y, z\} = \frac{1}{2} (\langle x, y \rangle z + \langle z, y \rangle x - \langle x, z^* \rangle y^*).$$

The only possible infinite dimensional finite-rank Cartan factors are the spin factors and  $L(\mathbb{C}^r, H)$ , with  $\dim H = \infty$ , where a spin factor has rank 2 and  $L(\mathbb{C}^r, H)$  has rank  $r$ . The open unit ball of a spin factor is known as a *Lie ball*. The open unit balls of the first four types of finite dimensional Cartan factors are the classical *Cartan domains*.

Let  $V$  be a JB\*-triple of finite rank  $r$ . Then the sum of  $r$  orthogonal minimal tripotents  $e_1, \dots, e_r$  is a maximal tripotent and  $V_{00} = V_0(e_1 + \dots + e_r) = \{0\}$  by (3.4). Each  $x \in V$  has a *spectral decomposition*

$$x = \alpha_1 e_1 + \dots + \alpha_r e_r \tag{4.1}$$

for some mutually orthogonal minimal tripotents  $e_1, \dots, e_r$ , where the uniquely determined coefficients satisfy  $0 \leq \alpha_r \leq \dots \leq \alpha_1$  with  $\alpha_1 = \|x\|$ .

Since a finite-rank JB\*-triple is reflexive, its open unit ball is relatively compact in the weak topology. We will exploit the weak topology in our computation in the infinite dimensional case, which involves the spin factors and the Type I Cartan factors  $L(\mathbb{C}^r, H)$ .

The minimal tripotents in  $L(\mathbb{C}^r, H)$  are exactly the rank-one operators  $a \otimes b : \mathbb{C}^r \rightarrow H$  with  $\|a\|_{\mathbb{C}^r} = \|b\|_H = 1$ , where

$$(a \otimes b)(\mu) = \langle \mu, a \rangle b \quad (\mu \in \mathbb{C}^r)$$

and the adjoint  $(a \otimes b)^*$  is the rank-one operator  $b \otimes a : H \rightarrow \mathbb{C}^r$  given by  $(b \otimes a)(h) = \langle h, b \rangle a$  for  $h \in H$ . We have used the same symbol  $\langle \cdot, \cdot \rangle$  for inner products in  $\mathbb{C}^r$  and  $H$ , which should not cause any confusion. For convenience, we write  $x \perp y$  to denote that  $x$  and  $y$  are orthogonal in a Hilbert space, not to be confused with the notion of orthogonality in a JB\*-triple.

**Lemma 4.1.** *Let  $e_i = a_i \otimes b_i$  be a rank-one operator in  $L(\mathbb{C}^r, H)$  for  $i = 1, 2$ . Then  $e_1 \square e_2 = 0$  if and only if  $a_1 \perp a_2$  and  $b_1 \perp b_2$  in their respective Hilbert spaces. In this case, we have  $e_1(\mu) \perp e_2(\mu)$  in  $H$  for all  $\mu \in \mathbb{C}^r$ .*

*Proof.* Let  $e_1 \square e_2 = 0$ . Then we have  $0 = \{e_1, e_2, f\} = \frac{1}{2}(e_1 e_2^* f + f e_2^* e_1)$  for all  $f \in L(\mathbb{C}^r, H)$ . More explicitly we have

$$\begin{aligned} 0 &= \langle \langle f(\cdot), b_2 \rangle a_2, a_1 \rangle b_1 + f[\langle \langle \cdot, a_1 \rangle b_1, b_2 \rangle a_2] \\ &= \langle f(\cdot), b_2 \rangle \langle a_2, a_1 \rangle b_1 + \langle \cdot, a_1 \rangle \langle b_1, b_2 \rangle f(a_2). \end{aligned}$$

In particular, when  $f = e_1$ , we have  $0 = \langle \cdot, a_1 \rangle \langle b_1, b_2 \rangle \langle a_2, a_1 \rangle b_1$ , which clearly implies either  $a_1 \perp a_2$  or  $b_1 \perp b_2$ . On the other hand, if  $f = e_2$ , then

$$\begin{aligned} 0 &= \langle \langle \cdot, a_2 \rangle b_2, b_2 \rangle \langle a_2, a_1 \rangle b_1 + \langle \cdot, a_1 \rangle \langle b_1, b_2 \rangle \langle a_2, a_2 \rangle b_2 \\ &= \langle \cdot, a_2 \rangle \|b_2\|^2 \langle a_2, a_1 \rangle b_1 + \langle \cdot, a_1 \rangle \langle b_1, b_2 \rangle \|a_2\|^2 b_2. \end{aligned}$$

This gives  $a_1 \perp a_2$  and  $b_1 \perp b_2$ . In this case, we also have  $\langle e_1(\mu), e_2(\mu) \rangle = \langle \langle \mu, a_1 \rangle b_1, \langle \mu, a_2 \rangle b_2 \rangle = \langle \mu, a_1 \rangle \overline{\langle \mu, a_2 \rangle} \langle b_1, b_2 \rangle = 0$ , for  $\mu \in \mathbb{C}^r$ .

Conversely, suppose  $a_1 \perp a_2$  and  $b_1 \perp b_2$ . Given any  $x \in L(\mathbb{C}^r, H)$ , we have

$$\begin{aligned} 2(e_1 \square e_2)(x) &= 2\{e_1, e_2, x\} = e_1 e_2^* x + x e_2^* e_1 \\ &= \langle x(\cdot), b_2 \rangle \langle a_2, a_1 \rangle b_1 + \langle \cdot, a_1 \rangle \langle b_1, b_2 \rangle x(a_2) = 0. \end{aligned}$$

□

Let  $D$  be a Lie ball, realised as the open unit ball of a spin factor  $V$ , equipped with an inner product  $\langle \cdot, \cdot \rangle$  and involution  $*$ . We now prove a convergence result for  $D$  in the following lemma which will be used later. This result also simplifies and improves the arguments showing the convergence of the sequence  $(c_k(y))$  in [8, p. 130-132].

To prove the lemma, we make use of the fact [26, Lemma 5.10] that for any two triple orthogonal elements  $u, v \in D$ , the Möbius transformation  $g_{u+v}$  satisfies

$$g_{u+v} = g_u \circ g_v.$$

**Lemma 4.2.** *Let  $(z_k)$  be a sequence in a Lie ball  $D$  norm converging to  $\xi \in \overline{D}$ , where*

$$z_k = \alpha_{1k} d_k + \alpha_{2k} d_k^*,$$

*$(d_k)$  is a sequence of minimal tripotents weakly converging to  $d \in \overline{D}$  and  $\alpha_{1k} = \|z_k\| \geq |\alpha_{2k}|$  with  $\alpha_{2k} \in \mathbb{C}$ .*

*If  $\xi \neq 0$ , then both sequences  $(d_k)$  and  $(d_k^*)$  are norm convergent, in which case  $d$  and  $d^*$  are minimal tripotents. If  $\xi = 0$ , then both sequences  $(\{d_k, z_k, d_k\})$  and  $(\{d_k^*, z_k, d_k^*\})$  norm converge to 0.*

*Proof.* We note that  $d_k \square d_k^* = \langle d_k, d_k^* \rangle = 0$  [8, Lemma 2.3] and  $\lim_k \alpha_{1k} = \|\xi\|$ . Also,  $\lim_k \alpha_{2k} = \alpha \in \overline{\mathbb{D}}$  with  $|\alpha| \leq \|\xi\|$ .

If  $\xi = 0$ , then  $\lim_k \alpha_{1k} = 0$  and  $\alpha = 0$ . Hence both sequences  $\{d_k, z_k, d_k\} = \alpha_{1k} d_k$  and  $\{d_k^*, z_k, d_k^*\} = \overline{\alpha_{2k}} d_k^*$  norm converge to 0.

Let  $\xi \neq 0$ . We show every subsequence of  $(d_k)$  contains a norm convergent subsequence which would complete the proof. To simplify notation, we pick a subsequence and still denote it by  $(d_k)$ .

We first consider the case  $\|\xi\| = 1$  in which situation, it has been shown in [8, p.126] that  $(d_k)$  is norm convergent if  $|\alpha| < 1$ . Hence we only need to show norm convergence for  $|\alpha| = 1$ . In this

case, the Bergmann operators  $B(z_k, z_k)$  norm converge to 0 by the formula

$$B(z_k, z_k) = (1 - \alpha_{1k}^2)^2 P_2(d_k) + (1 - \alpha_{1k}^2)(1 - |\alpha_{2k}|^2) P_1(d_k) + (1 - |\alpha_{2k}|^2)^2 P_0(d_k)$$

since the Peirce projections are contractive. Pick  $y \in D \setminus \mathbb{C}\xi$  and let  $w_k = g_{-z_k}(y)$ . Then

$$\lim_k w_k = \lim_k (-z_k + B(z_k, z_k)^{1/2}(\mathbf{1} - y \square z_k)^{-1}(y)) = -\xi.$$

Write  $g_{z_k}(w_k) = g_{\alpha_{1k}d_k}(g_{\alpha_{2k}d_k^*}(w_k))$  and  $x_k = g_{\alpha_{2k}d_k^*}(w_k)$ .

For each  $x \in D$ , we have

$$\begin{aligned} (x \square \alpha_{1k}d_k)(x) &= \alpha_{1k}\langle x, d_k \rangle x - \frac{\alpha_{1k}\langle x, x^* \rangle}{2} d_k^* \\ (x \square \alpha_{1k}d_k)^2(x) &= \alpha_{1k}^2\langle x, d_k \rangle^2 x - \frac{\alpha_{1k}^2\langle x, x^* \rangle \langle x, d_k \rangle}{2} d_k^* \\ (x \square \alpha_{1k}d_k)^3(x) &= \alpha_{1k}^3\langle x, d_k \rangle^3 x - \frac{\alpha_{1k}^3\langle x, x^* \rangle \langle x, d_k \rangle^2}{2} d_k^* \end{aligned}$$

and so on. Hence

$$\begin{aligned} (I + x \square d_k)^{-1}x &= (I - x \square d_k + (x \square d_k)^2 - \dots)(x) \\ &= (1 - \alpha_{1k}\langle x, d_k \rangle + \alpha_{1k}^2\langle x, d_k \rangle^2 - \dots)x + \frac{\alpha_{1k}\langle x, x^* \rangle}{2}(1 - \alpha_{1k}\langle x, d_k \rangle + \alpha_{1k}^2\langle x, d_k \rangle^2 - \dots)d_k^* \\ &= \frac{x}{1 + \alpha_{1k}\langle x, d_k \rangle} + \frac{\alpha_{1k}\langle x, x^* \rangle d_k^*}{2(1 + \alpha_{1k}\langle x, d_k \rangle)}. \end{aligned}$$

It follows from

$$B(\alpha_{1k}d_k, \alpha_{1k}d_k)^{1/2} = P_0(d_k) + \sqrt{1 - \alpha_{1k}^2}P_1(d_k) + (1 - \alpha_{1k}^2)P_2(d_k)$$

that

$$\begin{aligned} g_{\alpha_{1k}d_k}(x) &= \alpha_{1k}d_k + \frac{2\langle x, d_k^* \rangle + \alpha_{1k}\langle x, x^* \rangle}{2(1 + \alpha_{1k}\langle x, d_k \rangle)} d_k^* \\ &+ \sqrt{1 - \alpha_{1k}^2}P_1(d_k) \left( \frac{x}{1 + \alpha_{1k}\langle x, d_k \rangle} \right) + (1 - \alpha_{1k}^2)P_2(d_k) \left( \frac{x}{1 + \alpha_{1k}\langle x, d_k \rangle} \right). \end{aligned}$$

Likewise, we have

$$\begin{aligned} g_{\alpha_{2k}d_k^*}(x) &= \alpha_{2k}d_k^* + \frac{2\langle x, d_k \rangle + \bar{\alpha}_{2k}\langle x, x^* \rangle}{2(1 + \bar{\alpha}_{2k}\langle x, d_k^* \rangle)} d_k \\ &+ \sqrt{1 - |\alpha_{2k}|^2}P_1(d_k) \left( \frac{x}{1 + \bar{\alpha}_{2k}\langle x, d_k^* \rangle} \right) + (1 - |\alpha_{2k}|^2)P_0(d_k) \left( \frac{x}{1 + \bar{\alpha}_{2k}\langle x, d_k^* \rangle} \right). \end{aligned}$$

Therefore we have

$$\begin{aligned} g_{\alpha_{1k}d_k}(g_{\alpha_{2k}d_k^*}(w_k)) &= g_{\alpha_{1k}d_k}(x_k) = \alpha_{1k}d_k + \frac{2\langle x_k, d_k^* \rangle + \alpha_{1k}\langle x_k, x_k^* \rangle}{2(1 + \alpha_{1k}\langle x_k, d_k \rangle)} d_k^* \\ &+ \sqrt{1 - \alpha_{1k}^2}P_1(d_k) \left( \frac{x_k}{1 + \alpha_{1k}\langle x_k, d_k \rangle} \right) + (1 - \alpha_{1k}^2)P_2(d_k) \left( \frac{x_k}{1 + \alpha_{1k}\langle x_k, d_k \rangle} \right). \end{aligned}$$

Since

$$\begin{aligned} \left| \frac{\sqrt{1 - |\alpha_{2k}|^2}}{1 + \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle} \right|^2 &= \frac{1 - |\alpha_{2k}|^2}{1 + |\alpha_{2k}|^2 |\langle w_k, d_k^* \rangle|^2 + 2 \operatorname{Re} \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle} \\ &= 1 - \frac{|\alpha_{2k}|^2 + |\alpha_{2k}|^2 |\langle w_k, d_k^* \rangle|^2 + 2 \operatorname{Re} \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle}{1 + |\alpha_{2k}|^2 |\langle w_k, d_k^* \rangle|^2 + 2 \operatorname{Re} \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle} \leq 2, \end{aligned}$$

we may assume, by choosing a subsequence, that the complex sequence

$$\frac{\sqrt{1 - |\alpha_{2k}|^2}}{1 + \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle}$$

converges. Likewise, we may assume that the sequence

$$\frac{\sqrt{1 - \alpha_{1k}^2}}{1 + \alpha_{1k} \langle x_k, d_k \rangle}$$

converges.

Observe that

$$P_1(d_k)(x_k) = P_1(d_k)(g_{\alpha_{2k}d_k^*}(w_k)) = \frac{\sqrt{1 - |\alpha_{2k}|^2}}{1 + \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle} P_1(d_k)(w_k).$$

Writing

$$P_1(d_k)(w_k) = w_k - \langle w_k, d_k \rangle d_k - \langle w_k, d_k^* \rangle d_k^*,$$

it can be seen that

$$g_{z_k}(w_k) = g_{\alpha_{1k}d_k}(g_{\alpha_{2k}d_k^*}(w_k)) = \frac{\sqrt{1 - \alpha_{1k}^2} \sqrt{1 - |\alpha_{2k}|^2}}{(1 + \alpha_{1k} \langle x_k, d_k \rangle)(1 + \overline{\alpha_{2k}} \langle w_k, d_k^* \rangle)} w_k + A_k d_k + B_k d_k^*$$

for some  $A_k, B_k \in \mathbb{C}$ . From this we infer that the sequence  $(A_k d_k + B_k d_k^*)$  norm converges to  $y + \beta \xi$  for some  $\beta \in \mathbb{C}$ . Combining this norm convergence with the given norm convergence of the sequence  $(\alpha_{1k} d_k + \alpha_{2k} d_k^*)$  to  $\xi$  and, noting  $y \notin \mathbb{C}\xi$ , we conclude that the sequence  $(d_k)$  is norm convergent to  $d$ .

Finally, consider the case  $\|\xi\| < 1$ . Since  $\xi \neq 0$ , we may assume  $z_k \neq 0$  by omitting, if necessary, the first few terms of the sequence. Let

$$z'_k = \frac{z_k}{(1 + 1/2^k)\|z_k\|} = \frac{\alpha_{1k}}{(1 + 1/2^k)\|z_k\|} d_k + \frac{\alpha_{2k}}{(1 + 1/2^k)\|z_k\|} d_k^*.$$

Then  $(z'_k)$  is a sequence in  $D$  norm converging to  $\xi/\|\xi\|$ . By the previous case, one concludes with the norm convergence  $d = \lim_k d_k$ . This proves the first assertion.  $\square$

## 5. HOROBALLS IN FINITE RANK BOUNDED SYMMETRIC DOMAINS

In this section, we show that the invariant domains  $H(\xi, \lambda)$  of a fixed-point free compact holomorphic self-map  $f$  on a finite rank bounded symmetric domain are horoballs  $S_0(\xi, \lambda)$  resembling the horodiscs in Wolff's theorem for  $\mathbb{D}$ .

Throughout, let  $D$  be a finite-rank bounded symmetric domain, realised as the open unit ball of a JB\*-triple  $V$  of rank  $p$ , with a decomposition

$$V = V_1 \oplus \cdots \oplus V_q$$

into an  $\ell^\infty$ -sum of Cartan factors  $V_1, \dots, V_q$ , which are mutually orthogonal, that is,  $V_i \square V_j = \{0\}$  for distinct  $i, j \in \{1, \dots, q\}$ . For each  $z \in V$  in the sequel, we shall write the spectral decomposition of  $z$  in the following form

$$z = \alpha_1 e_1 + \dots + \alpha_p e_p \quad (\alpha_1, \dots, \alpha_p \geq 0) \quad (5.1)$$

where two consecutive minimal tripotents are either in the same direct summand, or belong to two consecutive summands, that is, there exist  $i \geq 1$  and  $1 \leq r < \dots < \ell$  such that

$$\{e_1, \dots, e_i\} \subset V_1, \{e_{i+1}, \dots, e_{i+r}\} \subset V_2, \dots, \{e_{i+\ell}, \dots, e_p\} \subset V_q. \quad (5.2)$$

Given a direct sum decomposition  $z = z_1 + \dots + z_q \in V_1 \oplus \dots \oplus V_q$ , triple orthogonality implies

$$\{e_k, z, e_k\} = \{e_k, z_j, e_k\} \quad (5.3)$$

if  $e_k$  belongs to the summand  $V_j$ . Considering each element  $z \in V$  having  $q$  summands, we note that norm and weak convergence of a sequence in  $V$  are the same as norm and weak convergence in each summand. In fact, the weak and norm topologies of  $V$  are product topologies of those of  $V_1, \dots, V_q$ .

We sometimes use the symbol  $x_k \xrightarrow{w} x$  to denote weak convergence of a sequence  $(x_k)$ . In the Cartan factor  $L(\mathbb{C}^r, H)$ , we have  $x_k \xrightarrow{w} x$  if and only if  $\langle x_k(\mu), h \rangle \rightarrow \langle x(\mu), h \rangle$  as  $k \rightarrow \infty$ , for all  $\mu \in \mathbb{C}^r$  and  $h \in H$ . The weak convergence  $x_k \xrightarrow{w} x$  implies  $\{x_k, a, x_k\} \xrightarrow{w} \{x, a, x\}$  for each  $a \in L(\mathbb{C}^r, H)$  since

$$\langle \{x_k, a, x_k\}(\mu), h \rangle = \langle x_k a^* x_k(\mu), h \rangle = \langle a^* x_k(\mu), x_k^*(h) \rangle \rightarrow \langle a^* x(\mu), x^*(h) \rangle$$

where weak and norm convergence are the same in  $\mathbb{C}^r$ .

**Lemma 5.1.** *Let  $V$  be a finite-rank  $JB^*$ -triple without a spin factor direct summand and  $(z_k)$  a sequence in  $V$  norm converging to some  $\xi \in V$ . Given a sequence  $(e_k)$  of minimal tripotents in  $V$  weakly converging to  $e \in V$ , we have the weak convergence  $\{e_k, z_k, e_k\} \xrightarrow{w} \{e, \xi, e\}$  and also,  $\{e, V, e\} \subset \mathbb{C}e$ .*

*Proof.* Let  $V$  be an  $\ell^\infty$ -sum  $V_1 \oplus \dots \oplus V_q$  of mutually orthogonal Cartan factors. If the assertion is true for each  $V_j$  ( $j = 1, \dots, q$ ), then it is also true for  $V$  by (5.3) and the subsequent remark there. Hence it suffices to prove the lemma for a Type I Cartan factor  $V = L(\mathbb{C}^r, H)$ .

The above remark allows us to assume  $\xi = 0$  and show  $\{e_k, z_k, e_k\} \xrightarrow{w} 0$ . Indeed, we have

$$\begin{aligned} |\langle e_k z_k^* e_k(\mu), h \rangle| &= |\langle z_k^* e_k(\mu), e_k^* h \rangle| \\ &\leq \|z_k^* e_k(\mu)\|_{\mathbb{C}^r} \cdot \|e_k^* h\|_{\mathbb{C}^r} \\ &\leq \|z_k^*\| \cdot \|e_k(\mu)\|_H \cdot \|e_k^*\| \cdot \|h\|_H \\ &\leq \|z_k\| \cdot \|\mu\|_{\mathbb{C}^r} \cdot \|h\|_H \rightarrow 0 \end{aligned}$$

for all  $\mu \in \mathbb{C}^r$  and  $h \in H$ .

For the second assertion, let  $z \in V$ . Then  $\{e, z, e\} = \text{weak-}\lim_k \{e_k, z, e_k\} = \text{weak-}\lim_k \lambda_k e_k$  for some  $\lambda_k \in \mathbb{C}$ . It follows that the sequence  $|\lambda_k| = \|\lambda_k e_k\|$  is bounded and there is a subsequence of  $(\lambda_k)$  converging to some  $\lambda \in \mathbb{C}$ . This gives  $\{e, z, e\} = \lambda e$ .  $\square$

**Lemma 5.2.** *Let  $V$  be a  $JB^*$ -triple of finite rank  $p$  and  $(z_k)$  a sequence in  $D$  norm converging to some  $\xi \in \overline{D}$ , with spectral decomposition*

$$z_k = \alpha_{1k} e_{1k} + \dots + \alpha_{pk} e_{pk}$$

*where each sequence  $(e_{ik})_k$  weakly converges to  $e_i$  for  $i = 1, \dots, p$ . Then the sequence  $(\{e_{ik}, z_k, e_{ik}\})_k$  weakly converges to  $\{e_i, \xi, e_i\}$  for each  $i$ .*

*Proof.* As noted before,  $V$  is a finite  $\ell^\infty$ -sum of mutually orthogonal Cartan factors and we need only consider convergence in each summand. If  $V$  does not have a spin factor summand, this has already been proven in Lemma 5.1. If  $V$  contains a spin factor summand with inner product  $\langle \cdot, \cdot \rangle$ , in which a spectral decomposition of an element  $z$  has the form

$$z = \alpha_1 d + \beta e = \alpha_1 d + \beta \langle d, e^* \rangle d^* \quad (0 \leq \beta \leq \alpha_1)$$

(cf. [8, (2.4)]) where the involution  $*$  preserves weak convergence, then one can use Lemma 4.2 to conclude the proof.  $\square$

**Lemma 5.3.** *Let  $(e_k)$  and  $(u_k)$  be two weakly convergent sequences of minimal tripotents in  $L(\mathbb{C}^r, H)$ , with limits  $e$  and  $u$  respectively. If  $e_k u_k^* = 0$  from some  $k$  onwards, then  $eu^* = 0$ .*

*Proof.* We have

$$e_k = a_k \otimes b_k \quad \text{and} \quad u_k = c_k \otimes d_k$$

for some  $a_k, c_k \in \mathbb{C}^r$  and  $b_k, d_k \in H$ , of unit norm.

Pick two subsequences  $(a_j)$  and  $(b_j)$  of  $(a_k)$  and  $(b_k)$  respectively such that  $a \in \mathbb{C}^r$  is the norm limit of  $(a_j)$  and  $b \in H$  is the weak limit of  $(b_j)$ . Then we have the weak convergence  $e = \text{weak-}\lim_j e_j = \text{weak-}\lim_j a_j \otimes b_j = a \otimes b$ . Likewise,  $u = c \otimes d$  where  $c$  is the norm limit of a subsequence  $(c_{j'})$  of  $(c_j)$ , and  $d$  the weak limit of a subsequence  $(d_{j'})$  of  $(d_j)$ .

Let  $\mu \in \mathbb{C}^r$  and  $h \in H$ . Then for each  $z \in L(\mathbb{C}^r, H)$ , we have

$$\begin{aligned} \langle eu^* z(\mu), h \rangle &= \langle z(\mu), d \rangle \langle c, a \rangle \langle b, h \rangle \\ &= \lim_{j'} \langle z(\mu), d_{j'} \rangle \langle c_{j'}, a_{j'} \rangle \langle b_{j'}, h \rangle \\ &= \lim_{j'} \langle e_{j'} u_{j'}^* z(\mu), h \rangle = 0 \end{aligned}$$

which implies  $eu^* = 0$ .  $\square$

**Corollary 5.4.** *Let  $V$  be a finite-rank  $JB^*$ -triple without a spin factor direct summand. Let  $(e_k)$  and  $(u_k)$  be weakly convergent sequences of minimal tripotents with limits  $e$  and  $u$  respectively such that  $e_k$  and  $u_k$  are orthogonal for each  $k$ . Then  $\{e, u, e\} = 0$ .*

*Proof.* We only need to verify the case where  $V$  is the Cartan factor  $L(\mathbb{C}^r, H)$  for some Hilbert space  $H$ . We retain the notation from the previous proof. As  $(e_k)$  and  $(u_k)$  are triple orthogonal, we have  $\langle c_k, a_k \rangle = 0$  by Lemma 4.1 and therefore  $e_k u_k^* = \langle \cdot, d_k \rangle \langle c_k, a_k \rangle b_k = 0$ . Hence Lemma 5.3 finishes the proof.  $\square$

**Lemma 5.5.** *Let  $V$  be a  $JB^*$ -triple of finite rank  $p$  and  $(z_k)$  a sequence in  $D$  norm converging to some  $\xi \in \overline{D}$ , with spectral decomposition*

$$z_k = \alpha_{1k} e_{1k} + \cdots + \alpha_{pk} e_{pk} \quad (\alpha_{1k}, \dots, \alpha_{pk} \geq 0)$$

*where each sequence  $(e_{jk})_k$  weakly converges to  $e_j$  and  $(\alpha_{jk})_k$  converges to  $\alpha_j$ , for  $j = 1, \dots, p$ . Then we have  $\alpha_j \{e_i, e_j, e_i\} = 0$  for  $i, j \in \{1, \dots, p\}$  and  $i \neq j$ .*

*Proof.* If  $e_i$  and  $e_j$  do not belong to the same direct summand then  $\{e_i, e_j, e_i\} = 0$ . If they do, then the result follows from Lemma 4.2 and Corollary 5.4, by considering each summand in the decomposition of  $V$  into Cartan factors.  $\square$

**Lemma 5.6.** *Let  $V$  be a finite-rank  $JB^*$ -triple without a spin factor direct summand. Then a weakly convergent sequence  $(e_k)$  of minimal tripotents in  $V$  with a minimal tripotent limit is norm convergent.*



*Proof.* We only need to consider the case where  $V$  is a finite  $\ell^\infty$ -sum  $V_1 \oplus \cdots \oplus V_q$  of Type I Cartan factors. Let  $c = (c_1, \dots, c_q) \in V$  be the weak limit of the sequence  $(e_k)$  and write  $e_k = (e_{1k}, \dots, e_{qk})$  which has only one nonzero coordinate. Since  $c$  is a minimal tripotent, there exists  $j \in \{1, \dots, q\}$  such that  $c_j$  is a minimal tripotent in  $V_j$  and  $c_i = 0$  for all  $i \neq j$ . By coordinatewise weak convergence, there exists  $K$  such that  $k \geq K$  implies  $e_{ik} = 0$  for  $i \neq j$ , and  $e_{jk}$  is a minimal tripotent in  $V_j$ .

Let  $V_j = L(\mathbb{C}^r, H)$  for some Hilbert space  $H$  in which case, we have

$$e_{jk} = a_k \otimes b_k \quad (a_k \in \mathbb{C}^r, b_k \in H)$$

and  $\|a_k\| = \|b_k\| = 1$ .

To complete the proof, we show that every subsequence of  $(e_{jk})$  has a subsequence norm converging to  $c_j$ . Let  $(a_{k'} \otimes b_{k'})$  be a subsequence of  $(e_{jk})$ . We can find a subsequence  $(a_{k''} \otimes b_{k''})$  of  $(a_{k'} \otimes b_{k'})$  which weakly converges to  $a'' \otimes b''$  with  $a'' = \lim_{k''} a_{k''}$  and  $b'' = \text{weak-lim}_{k''} b_{k''}$ . It follows that  $c_j = a'' \otimes b''$ , which implies  $\|b''\| = 1$  and  $(b_{k''})$  actually norm converges to  $b''$  in the Hilbert space  $H$ . A simple calculation then shows that  $(a_{k''} \otimes b_{k''})$  norm converges to  $c_j$ .  $\square$

Now let  $(z_k)$  be a sequence in  $D$  norm converging to some  $\xi \in \partial D$ . As in (5.1) and (5.2), choose a spectral decomposition of each  $z_k$ :

$$z_k = \alpha_{1k}e_{1k} + \cdots + \alpha_{pk}e_{pk}, \quad (5.4)$$

where  $\alpha_{1k}, \dots, \alpha_{pk} \geq 0$  and  $\{e_{1k}, \dots, e_{pk}\}$  is a family of mutually orthogonal minimal tripotents in  $\partial D$ . By weak compactness of  $\overline{D}$ , each sequence  $(e_{ik})_k$  has a weak limit point  $e_i$  say, for  $i = 1, \dots, p$ . Replace  $(z_k)$  by a subsequence if necessary, we may assume henceforth that  $(\alpha_{ik})$  converges to  $\alpha_i$ , and that  $(e_{ik})$  weakly converges to  $e_i$  for  $i = 1, \dots, p$ . It follows that

$$\xi = \alpha_1 e_1 + \cdots + \alpha_p e_p.$$

**Lemma 5.7.** *Let  $V$  be a  $JB^*$ -triple of finite rank  $p$  and  $(z_k)$  be the sequence norm converging to  $\xi$  as defined above. Then there exists a non-empty set  $J \subset \{1, \dots, p\}$  such that*

- (i)  $\alpha_i \neq 0$  and  $(e_{ik})_k$  norm converges to  $e_i$ , for each  $i \in J$ ;
- (ii)  $\{e_i : i \in J\}$  is a family of pairwise orthogonal minimal tripotents;
- (iii)  $\alpha_i = 0$  and  $(\alpha_{ik}e_{ik})$  norm converges to 0, for each  $i \in \{1, \dots, p\} \setminus J$ .

*Proof.* For  $i \in \{1, \dots, p\}$ , we have

$$\{e_{ik}, z_k, e_{ik}\} = \sum_{j=1}^p \alpha_{jk} \{e_{ik}, e_{jk}, e_{ik}\} = \alpha_{ik} e_{ik}$$

which converges weakly to both  $\alpha_i e_i$  and, by Lemma 5.2, to

$$\{e_i, \xi, e_i\} = \sum_{j=1}^p \alpha_j \{e_i, e_j, e_i\} = \alpha_i \{e_i, e_i, e_i\} + \sum_{j \neq i} \alpha_j \{e_i, e_j, e_i\}.$$

This, together with Lemma 5.5, gives

$$\alpha_i e_i = \alpha_i \{e_i, e_i, e_i\} + \sum_{j \neq i} \alpha_j \{e_i, e_j, e_i\} = \alpha_i \{e_i, e_i, e_i\}.$$

Therefore either  $\alpha_i = 0$  or  $e_i$  is a tripotent. If  $\alpha_i e_i \neq 0$ , then Lemma 4.2 and Lemma 5.1 imply that  $e_i$  is a minimal tripotent and also, it follows from Lemma 4.2 and Lemma 5.6 that the sequence

$(e_{ik})$  actually norm converges to  $e_i$ . Let

$$J = \{i \in \{1, \dots, p\} : \alpha_i e_i \neq 0\}.$$

Then  $J \neq \emptyset$  as  $\sum_{i=1}^p \alpha_i e_i = \xi \neq 0$ .

For each  $i \in J$ , we have  $\alpha_i \neq 0$  and by norm convergence of  $(e_{ik})$ , the minimal tripotents  $\{e_i : i \in J\}$  are pairwise orthogonal. This proves (i) and (ii).

To show (iii), let  $j \in \{1, \dots, p\} \setminus J$ . Then we have the norm convergence

$$\begin{aligned} \alpha_{jk} = \|\alpha_{jk} e_{jk}\| &\leq \max\{\|\alpha_{ik} e_{ik}\| : i \in \{1, \dots, p\} \setminus J\} = \left\| \sum_{i \in \{1, \dots, p\} \setminus J} \alpha_{ik} e_{ik} \right\| \\ &= \|z_k - \sum_{i \in J} \alpha_{ik} e_{ik}\| \longrightarrow \|\xi - \sum_{i \in J} \alpha_i e_i\| = 0 \end{aligned}$$

which gives  $\alpha_j = 0$ . The second assertion in (iii) follows from  $\|e_{ik}\| = 1$ .  $\square$

*Remark 5.8.* Let  $V$  be a finite-rank JB\*-triple and let  $(z_k)$ ,  $\xi$  and  $J$  be as defined in Lemma 5.7. The norm convergence of the sequence  $(e_{ik})_k$  to  $e_i$  for all  $i \in J$  and the pairwise orthogonality of the minimal tripotents  $\{e_i : i \in J\}$  ensure the norm convergence

$$\lim_{k \rightarrow \infty} P_{ij}(e_{sk} : s \in J) = P_{ij}(e_s : s \in J)$$

of a sequence of joint Peirce projections, for  $i, j \in \{0\} \cup J$ . Moreover, if  $(w_k)$  is a sequence in  $\overline{D}$  weakly converging to  $w \in \overline{D}$ , then we also have  $P_{ij}(e_{sk} : s \in J)(w_k) \xrightarrow{w} P_{ij}(e_s : s \in J)(w)$  for  $i, j \in \{0\} \cup J$ .

Let  $P_{jj'}^k$  denote the joint Peirce projections  $P_{jj'}(e_{1k}, \dots, e_{pk})$  for  $0 \leq j \leq j' \leq p$ . By (3.1) and (3.9), we have

$$\begin{aligned} (1 - r_k^2)B(r_k z_k, r_k z_k)^{-1/2}(z_k) &= (1 - r_k^2) \sum_{0 \leq j \leq j' \leq p} (1 - r_k^2 \alpha_{jk}^2)^{-1/2} (1 - r_k^2 \alpha_{j'k}^2)^{-1/2} P_{jj'}^k(z_k) \\ &= \sum_{j=1}^p \frac{1 - r_k^2}{1 - r_k^2 \alpha_{jk}^2} \alpha_{jk} e_{jk} \end{aligned}$$

where  $\alpha_{0k}$  is defined to be 0 for all  $k$ . We can write

$$\frac{1 - r_k^2}{1 - r_k^2 \alpha_{jk}^2} = \frac{\frac{1 - r_k^2}{1 - \|z_k\|^2}}{\frac{1 - r_k^2}{1 - \|z_k\|^2} + \left(\frac{1 - \alpha_{jk}^2}{1 - \|z_k\|^2}\right) r_k^2} = \frac{\left(\frac{1 - \|z_k\|^2}{1 - \alpha_{jk}^2}\right) \lambda}{\left(\frac{1 - \|z_k\|^2}{1 - \alpha_{jk}^2}\right) \lambda + r_k^2} \quad (5.5)$$

where  $\lambda(1 - \|z_k\|^2) = 1 - r_k^2$  and  $\frac{1 - \|z_k\|^2}{1 - \alpha_{jk}^2} \in (0, 1]$ . Let

$$\sigma_j := \limsup_k \frac{1 - \|z_k\|^2}{1 - \alpha_{jk}^2} \in [0, 1] \quad (j = 1, \dots, p). \quad (5.6)$$

The sequence in (5.5) admits a convergent subsequence

$$\frac{1 - r_{k'}^2}{1 - r_{k'}^2 \alpha_{jk'}^2} \longrightarrow \frac{\sigma_j \lambda}{\sigma_j \lambda + 1} \quad \text{as } k' \rightarrow \infty. \quad (5.7)$$

Noting that  $\alpha_j \in [0, 1]$  where  $\alpha_j < 1$  implies  $\sigma_j = 0$ , and  $\alpha_j = 0$  for  $j \in \{1, \dots, p\} \setminus J$ , we have the following norm convergence:

$$(1 - r_{k'}^2)B(r_{k'}z_{k'}, r_{k'}z_{k'})^{-1/2}(z_{k'}) \longrightarrow \sum_{j=1}^p \frac{\sigma_j \lambda}{\sigma_j \lambda + 1} \alpha_j e_j = \sum_{j \in J} \frac{\sigma_j \lambda}{\sigma_j \lambda + 1} e_j. \quad (5.8)$$

*Remark 5.9.* Since  $\|z_k\| = \max\{\alpha_{jk} : j = 1, \dots, p\}$ , there are two possibilities for each  $j \in \{1, \dots, p\}$ , namely, either  $\alpha_{jk} < \|z_k\|$  from some  $k$  onwards or there is a subsequence  $(\alpha_{jk'})_{k'}$  of  $(\alpha_{jk})_k$  with  $\alpha_{jk'} = \|z_{k'}\|$ . As  $\{1, \dots, p\}$  is finite, there exists some  $j_0$  which satisfies the latter in which case  $\sigma_{j_0} = 1$  and  $\alpha_{j_0} = 1$ .

We are now ready to give an explicit description of the closed horoball  $S(\xi, \lambda)$  in terms of a Bergmann operator.

**Theorem 5.10.** *Let  $D$  be a bounded symmetric domain of finite rank  $p$  and  $f : D \rightarrow D$  a fixed-point free compact holomorphic map. Then there exist a non-empty set  $J \subset \{1, \dots, p\}$  and a sequence  $(z_k)$  in  $D$  converging to a boundary point*

$$\xi = \sum_{j \in J} \alpha_j e_j \quad (0 < \alpha_j \leq 1)$$

where  $\{e_j : j \in J\}$  consists of mutually orthogonal minimal tripotents in  $\partial D$ , such that for each  $\lambda > 0$ ,

$$S(\xi, \lambda) = \sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (\overline{D})$$

with  $\sigma_j \geq 0$  and  $\max\{\sigma_j : j \in J\} = 1$ .

*Proof.* Let  $D$  be realised as the open unit ball of a JB\*-triple  $V$  of rank  $p$ . Let  $(z_k)$  be the sequence used in the construction of  $S(\xi, \lambda)$  in (2.8), where  $\lim_k z_k = \xi$  and as in (5.4), we fix a spectral decomposition

$$z_k = \alpha_{1k} e_{1k} + \dots + \alpha_{pk} e_{pk}$$

with  $\alpha_j = \lim_k \alpha_{jk}$  for  $j = 1, \dots, p$ . Throughout the proof  $\sigma_j$  is defined as in (5.6) and we take Remark 5.9 into account.

By Lemma 5.7, there exists a nonempty set  $J \subset \{1, \dots, p\}$  such that  $(e_{jk})_k$  norm converges to a minimal tripotent  $e_j$  for all  $j \in J$ , the tripotents  $\{e_s : s \in J\}$  are mutually orthogonal and  $\alpha_j = 0$  for  $j \in \{1, \dots, p\} \setminus J$ . We denote the latter set by  $J^c$  to simplify notation.

Let  $x \in S(\xi, \lambda)$ . Then we have  $x = \lim_k x_k$ , where  $x_k$  is an element in the Kobayashi ball  $D_k(\lambda)$  defined in (2.5). By (2.7), each  $x_k$  has the form

$$x_k = c_k + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2} (w_k)$$

where  $c_k = (1 - r_k^2)B(r_k z_k, r_k z_k)^{-1/2}(z_k)$  and  $w_k \in D$ .

To compute the norm limit  $\lim_k x_k$ , it suffices to compute a weak subsequential limit  $\lim_{k'} x_{k'}$ . By weak compactness and by (5.8), we may assume, by choosing subsequences if necessary, that  $(w_k)$  weakly converges to some  $w \in \overline{D}$  and that

$$\sigma_j = \lim_k \frac{1 - \|z_k\|^2}{1 - \alpha_{jk}^2}, \quad \lim_k c_k = \sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j. \quad (5.9)$$

By the formulae for the square roots of the Bergmann operator in (3.8) and (3.9), we have

$$\begin{aligned}
& r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2} (w_k) \\
&= r_k B(z_k, z_k)^{1/2} \sum_{0 \leq j \leq j' \leq p} (1 - r_k^2 \alpha_{jk}^2)^{-1/2} (1 - r_k^2 \alpha_{j'k}^2)^{-1/2} P_{jj'}(e_{1k}, \dots, e_{pk})(w_k) \\
&= r_k \sum_{0 \leq j \leq j' \leq p} \sqrt{\frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2}} \sqrt{\frac{1 - \alpha_{j'k}^2}{1 - r_k^2 \alpha_{j'k}^2}} P_{jj'}(e_{1k}, \dots, e_{pk})(w_k)
\end{aligned} \tag{5.10}$$

where  $\alpha_{0k} = 0$  for all  $k$ . Note that  $\alpha_j = 0$  implies  $\lim_k \frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2} = 1$  and also,  $\sigma_j = 0$ .

To compute the sum in (5.10), we split it into 3 summands, over the following 3 sets of indices:

$$\begin{aligned}
I &= \{0 \leq j \leq j' \leq p : j, j' \in \{0\} \cup J^c\}, \\
II &= \{(j, j') : j \in \{0\} \cup J^c, j' \in J\}, \\
III &= \{0 \leq j \leq j' \leq p : j, j' \in J\}.
\end{aligned}$$

We can write the first summand as follows.

$$\begin{aligned}
\sum_I &= \sum_{\substack{j \leq j' \\ j, j' \in \{0\} \cup J^c}} P_{jj'}(e_{1k}, \dots, e_{pk})(w_k) \\
&\quad - \sum_{\substack{j \leq j' \\ j, j' \in \{0\} \cup J^c}} \left( 1 - \sqrt{\frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2}} \sqrt{\frac{1 - \alpha_{j'k}^2}{1 - r_k^2 \alpha_{j'k}^2}} \right) P_{jj'}(e_{1k}, \dots, e_{pk})(w_k).
\end{aligned}$$

By Lemma 3.1(iii), we have

$$\begin{aligned}
\sum_I &= P_{00}(e_{sk} : s \in J)(w_k) \\
&\quad - \sum_{\substack{j \leq j' \\ j, j' \in \{0\} \cup J^c}} \left( 1 - \sqrt{\frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2}} \sqrt{\frac{1 - \alpha_{j'k}^2}{1 - r_k^2 \alpha_{j'k}^2}} \right) P_{jj'}(e_{1k}, \dots, e_{pk})(w_k).
\end{aligned}$$

For the second summand, Lemma 3.1(ii) enables us to write

$$\begin{aligned}
\sum_{II} &= \sum_{j' \in J} \sqrt{\frac{1}{r_k^2} \left( 1 - \frac{1 - r_k^2}{1 - r_k^2 \alpha_{j'k}^2} \right)} \left( \sum_{j \in \{0\} \cup J^c} P_{jj'}(e_{1k}, \dots, e_{pk})(w_k) \right. \\
&\quad \left. - \sum_{j \in \{0\} \cup J^c} \left( 1 - \sqrt{\frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2}} \right) P_{jj'}(e_{1k}, \dots, e_{pk})(w_k) \right) \\
&= \sum_{j' \in J} \sqrt{\frac{1}{r_k^2} \left( 1 - \frac{1 - r_k^2}{1 - r_k^2 \alpha_{j'k}^2} \right)} \left( P_{0j'}(e_{sk} : s \in J)(w_k) \right. \\
&\quad \left. - \sum_{j \in \{0\} \cup J^c} \left( 1 - \sqrt{\frac{1 - \alpha_{jk}^2}{1 - r_k^2 \alpha_{jk}^2}} \right) P_{jj'}(e_{1k}, \dots, e_{pk})(w_k) \right).
\end{aligned}$$

By Lemma 3.1(i), the third summand can be written as,

$$\sum_{III} = \sum_{\substack{j \leq j' \\ j, j' \in J}} \sqrt{\frac{1}{r_k^2} \left(1 - \frac{1 - r_k^2}{1 - r_k^2 \alpha_{jk}^2}\right)} \sqrt{\frac{1}{r_k^2} \left(1 - \frac{1 - r_k^2}{1 - r_k^2 \alpha_{j'k}^2}\right)} P_{jj'}(e_{sk} : s \in J)(w_k).$$

Noting that all Peirce projections are contractive and by Remark 5.8 as well as (3.8), we have the weak convergence

$$\begin{aligned} & \text{weak-}\lim_k r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(w_k) = \text{weak-}\lim_k r_k \left( \sum_I + \sum_{II} + \sum_{III} \right) \\ &= P_{00}(e_s : s \in J)(w) + \sum_{j' \in J} \sqrt{1 - \frac{\sigma_{j'} \lambda}{1 + \sigma_{j'} \lambda}} P_{0j'}(e_s : s \in J)(w) \\ &+ \sum_{\substack{j \leq j' \\ j, j' \in J}} \sqrt{1 - \frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} \sqrt{1 - \frac{\sigma_{j'} \lambda}{1 + \sigma_{j'} \lambda}} P_{jj'}(e_s : s \in J)(w) \\ &= B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (w). \end{aligned}$$

Now, together with (5.9), we have

$$x = \lim_k x_k = \sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (w)$$

which belongs to  $\sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (\overline{D})$ .

Conversely, let

$$y \in \sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (\overline{D}).$$

We show  $y \in S(\xi, \lambda)$ . There exists some  $x \in \overline{D}$  such that

$$y = \sum_{j \in J} \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j \in J} \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (x).$$

Let

$$y_k = (1 - r_k^2) B(r_k z_k, r_k z_k)^{-1/2}(z_k) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(x).$$

Then  $y_k \in D_k(\lambda)$ . Repeating the convergence arguments as before, with  $x$  in place of  $w_k$ , but with norm convergence as opposed to weak convergence, one sees that  $y = \lim_k y_k \in S(\xi, \lambda)$ , which completes the proof.  $\square$

*Remark 5.11.* By re-ordering the index set  $J$  in Theorem 5.10, the description of the boundary point  $\xi$  and  $S(\xi, \lambda)$  can be reformulated more simply as:

$$\xi = \sum_{j=1}^m \alpha_j e_j \quad (\alpha_j > 0)$$

for some  $m \in \{1, \dots, p\}$  and

$$S(\xi, \lambda) = \sum_{j=1}^m \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (\overline{D})$$

where  $\sigma_j \geq 0$  and  $\max\{\sigma_j : j = 1, \dots, m\} = 1$ .

For finite rank bounded symmetric domains, we now have the following generalisation of Wolff's theorem.

**Theorem 5.12.** *Let  $f$  be a fixed-point free compact holomorphic self-map on a bounded symmetric domain  $D$  of finite rank  $p$ . Then there is a sequence  $(z_k)$  in  $D$  converging to a boundary point*

$$\xi = \sum_{j=1}^m \alpha_j e_j \quad (\alpha_j > 0, m \in \{1, \dots, p\})$$

where  $e_1, \dots, e_m$  are orthogonal minimal tripotents in  $\partial D$ , such that for each  $\lambda > 0$ , the convex  $f$ -invariant domain  $H(\xi, \lambda)$  is the horoball  $S_0(\xi, \lambda)$  at  $\xi$ , which has the form

$$S_0(\xi, \lambda) = \sum_{j=1}^m \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (D) \quad (5.11)$$

and is affinely homeomorphic to  $D$ , where  $\sigma_j \geq 0$  and  $\max\{\sigma_j : j = 1, \dots, m\} = 1$ .

*Proof.* As before, we identify  $D$  as the open unit ball of a JB\*-triple  $V$  of rank  $p$ . Let  $(z_k)$  be the sequence in Theorem 5.10 with limit  $\xi = \sum_{j=1}^m \alpha_j e_j$  as in Remark 5.11. Let  $\{\sigma_j : j = 1, \dots, m\}$  be the same as in Remark 5.11. Observe that the map  $\varphi : V \rightarrow V$  defined by

$$\varphi = \sum_{j=1}^m \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2}$$

is an affine homeomorphism since the Bergmann operator  $B(a, a)^{1/2}$  is invertible whenever  $\|a\| < 1$ . Hence  $\varphi(D)$  is the interior of  $\varphi(\overline{D}) = S(\xi, \lambda)$ , that is,  $S_0(\xi, \lambda) = \varphi(D)$  which proves (5.11). It also shows that  $S_0(\xi, \lambda)$  and  $D$  are affinely homeomorphic. The equality  $H(\xi, \lambda) = S_0(\xi, \lambda)$  is shown in the next theorem.  $\square$

*Remark 5.13.* For  $p = 1$ , that is,  $D$  is a Hilbert ball, in the above theorem, we have

$$S_0(\xi, \lambda) = \frac{\lambda}{1 + \lambda} \xi + B \left( \sqrt{\frac{\lambda}{1 + \lambda}} \xi, \sqrt{\frac{\lambda}{1 + \lambda}} \xi \right)^{1/2} (D)$$

and in one dimension, that is,  $D = \mathbb{D}$ , it reduces to Wolff's horodisc in (2.1):

$$S_0(\xi, \lambda) = \frac{\lambda}{1 + \lambda} \xi + \frac{1}{1 + \lambda} \mathbb{D}.$$

**Theorem 5.14.** *Let  $f$  be a fixed-point free compact holomorphic self-map on a finite-rank bounded symmetric domain  $D$ . For each  $\lambda > 0$ , let  $H(\xi, \lambda)$  be the  $f$ -invariant domain defined in Theorem 2.4. Then we have  $H(\xi, \lambda) = S_0(\xi, \lambda)$  and the closure  $\overline{H}(\xi, \lambda)$  satisfies*

$$\overline{H}(\xi, \lambda) \cap D = S(\xi, \lambda) \cap D = \left\{ x \in D : \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} \leq \frac{1}{\lambda} \right\}.$$

*Proof.* By Lemma 2.6, it suffices to show  $S_0(\xi, \lambda) \subset H(\xi, \lambda)$  for the first assertion. By (5.11), we have

$$S_0(\xi, \lambda) = c(\lambda) + B(D)$$

where

$$c(\lambda) = \sum_{i=1}^m \frac{\sigma_i \lambda}{1 + \sigma_i \lambda} e_i \quad \text{and} \quad B = B \left( \sum_{i=1}^m \sqrt{\frac{\sigma_i \lambda}{1 + \sigma_i \lambda}} e_i, \sum_{i \in J} \sqrt{\frac{\sigma_i \lambda}{1 + \sigma_i \lambda}} e_i \right)^{1/2}.$$

Let  $x \in S_0(\xi, \lambda)$ . Then there exists  $v \in D$  such that

$$\begin{aligned} x &= c(\lambda) + B(v) \\ &= \lim_k [(1 - r_k^2)B(r_k z_k, r_k z_k)^{-1/2}(z_k) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(v)]. \end{aligned}$$

Let  $x_k = (1 - r_k^2)B(r_k z_k, r_k z_k)^{-1/2}(z_k) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(v)$ . Then by (2.6) we have

$$v = g_{r_k z_k} \left( \frac{g_{-z_k}(x_k)}{r_k} \right)$$

which gives  $g_{-z_k}(x_k) = r_k g_{-r_k z_k}(v)$  and

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x_k)\|^2} = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - r_k^2 \|g_{-r_k z_k}(v)\|^2}$$

where, by choosing a subsequence, we may replace the *upper limit*  $\limsup_k$  by the *limit*  $\lim_k$  in the following computation. Since, by (2.4),

$$\|g_{-r_k z_k}(v)\|^2 = 1 - \frac{1}{\|B(v, v)^{-1/2} B(v, r_k z_k) B(r_k z_k, r_k z_k)^{-1/2}\|},$$

we have

$$\begin{aligned} \frac{1 - r_k^2}{1 - \|g_{-z_k}(x_k)\|^2} &= \frac{1 - r_k^2}{1 - \|g_{-r_k z_k}(v)\|^2 + (1 - r_k^2) \|g_{-r_k z_k}(v)\|^2} \\ &= \frac{(1 - r_k^2) \|B(v, v)^{-1/2} B(v, r_k z_k) B(r_k z_k, r_k z_k)^{-1/2}\|}{r_k^2 + (1 - r_k^2) \|B(v, v)^{-1/2} B(v, r_k z_k) B(r_k z_k, r_k z_k)^{-1/2}\|}. \end{aligned} \quad (5.12)$$

We observe that the sequence  $((1 - r_k^2) \|B(v, v)^{-1/2} B(v, r_k z_k) B(r_k z_k, r_k z_k)^{-1/2}\|)$  is bounded by

$$\frac{(1 - r_k^2) \|B(v, v)^{-1/2}\| (1 + r_k \|v\| \|z_k\|)^2}{1 - r_k^2 \|z_k\|^2} \leq \|B(v, v)^{-1/2}\| (1 + \|v\|)^2.$$

Passing to a subsequence, we may assume that the sequence converges to some limit  $\ell \geq 0$ .

Hence we have, from (5.12),

$$\lim_{k \rightarrow \infty} \frac{1 - r_k^2}{1 - \|g_{-z_k}(x_k)\|^2} = \frac{\ell}{1 + \ell} < 1.$$

It follows that

$$\lim_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} = \lim_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x_k)\|^2} = \lim_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - r_k^2} \frac{1 - r_k^2}{1 - \|g_{-z_k}(x_k)\|^2} = \left(\frac{1}{\lambda}\right) \left(\frac{\ell}{1 + \ell}\right) < \frac{1}{\lambda}$$

and  $x \in H(\xi, \lambda)$ . This proves  $S_0(\xi, \lambda) \subset H(\xi, \lambda)$ .

For the second assertion, we need only show the last equality. It has already been shown in Lemma 2.6 that  $S(\xi, \lambda) \cap D \subset \{x \in D : F(x) \leq 1/\lambda\}$ . Conversely, let  $x \in D$  and  $F(x) \leq 1/\lambda$ . Then for  $0 < \varepsilon < 1$ , we have  $F(x) < \frac{1}{\varepsilon\lambda}$  which implies  $x \in H(\xi, \varepsilon\lambda) \subset S_0(\xi, \varepsilon\lambda)$ . Hence by (5.11),

$$x = \sum_{j=1}^m \frac{\sigma_j \varepsilon \lambda}{1 + \sigma_j \varepsilon \lambda} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j \varepsilon \lambda}{1 + \sigma_j \varepsilon \lambda}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j \varepsilon \lambda}{1 + \sigma_j \varepsilon \lambda}} e_j \right)^{1/2} (x_\varepsilon)$$

for some  $x_\varepsilon \in D$ . Let  $x_1 \in \overline{D}$  be a weak limit point of  $\{x_\varepsilon : 0 < \varepsilon < 1\}$ , and let  $\varepsilon \rightarrow 1$ . Then we have

$$x = \sum_{j=1}^m \frac{\sigma_j \lambda}{1 + \sigma_j \lambda} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j \lambda}{1 + \sigma_j \lambda}} e_j \right)^{1/2} (x_1) \in S(\xi, \lambda).$$

This completes the proof. □

The previous two results show that  $H(\xi, \lambda) \neq \emptyset$  for all  $\lambda > 0$  in finite rank bounded symmetric domains. We conclude this section by showing that this is in fact the case for *all* bounded symmetric domains. Let  $D$  be a bounded symmetric domain realised as the open unit ball of a JB\*-triple  $V$ . By the Gelfand-Naimark theorem for JB\*-triples [13],  $V$  can be realised as a closed subtriple of an  $\ell^\infty$ -sum  $\bigoplus_\iota V_\iota$  of Cartan factors  $V_\iota$ .

Given  $r > 0$ , we show that  $F^{-1}[0, r) \neq \emptyset$  for the function  $F$  in Lemma 2.1, which would entail  $H(\xi, 1/r) \neq \emptyset$ . For this, we need to find an element  $x \in D$  satisfying

$$\limsup_k \|B(x, x)^{-1/2} B(x, z_k) B(z_k, z_k)^{-1/2}\| (1 - \|z_k\|^2) < r$$

where the sequence  $(z_k)$  is as before, with limit  $\xi \in \partial D$ . We first observe that, given  $a \in V$ , the Bergmann operators  $B(a, a)$  with respect to  $V$  and with respect to  $\bigoplus_\iota V_\iota$  coincide on  $V$ . For our purpose, we may therefore assume  $V = \bigoplus_\iota V_\iota$  without loss of generality. Further, since the triple product on  $\bigoplus_\iota V_\iota$  is defined coordinatewise, we have, for  $a = \bigoplus_\iota a_\iota \in \bigoplus_\iota V_\iota$ ,

$$B(a, a) = \bigoplus_\iota B(a_\iota, a_\iota)$$

and it can be seen that, for  $x = \bigoplus_\iota x_\iota$  and  $z = \bigoplus_\iota z_\iota$  in  $V$ , we have

$$\begin{aligned} & \|B(x, x)^{-1/2} B(x, z) B(z, z)^{-1/2}\| (1 - \|z\|^2) \\ &= \sup_\iota \|B(x_\iota, x_\iota)^{-1/2} B(x_\iota, z_\iota) B(z_\iota, z_\iota)^{-1/2}\| (1 - \|z\|^2) \end{aligned} \quad (5.13)$$

$$\leq \sup_\iota \|B(x_\iota, x_\iota)^{-1/2} B(x_\iota, z_\iota) B(z_\iota, z_\iota)^{-1/2}\| (1 - \|z_\iota\|^2). \quad (5.14)$$

**Corollary 5.15.** *Let  $f$  be a fixed-point free compact holomorphic self-map on a bounded symmetric domain  $D$ . Then the  $f$ -invariant domain  $H(\xi, \lambda)$  is non-empty for each  $\lambda > 0$ .*



*Proof.* In view of previous remarks, we need only consider the case where  $D$  is the open unit ball of an  $\ell^\infty$ -sum  $V = \bigoplus_\iota V_\iota$  of Cartan factors. As in Lemma 2.1, let

$$F(x) = \limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x)\|^2} \quad (x \in D).$$

For each  $r > 0$ , we show  $F^{-1}[0, r) \neq \emptyset$  which would complete the proof. Let  $\xi = \bigoplus_\iota \xi_\iota$  where  $1 = \|\xi\| = \sup_\iota \|\xi_\iota\|$ . Since

$$F(x) = \limsup_k \|B(x, x)^{-1/2} B(x, z_k) B(z_k, z_k)^{-1/2}\| (1 - \|z_k\|^2),$$

using (5.13), it suffices to show that for each Cartan factor  $V_\iota$ , there is some  $x_\iota$  in its open unit ball  $D_\iota$  satisfying

$$\limsup_k \|B(x_\iota, x_\iota)^{-1/2} B(x_\iota, z_{k,\iota}) B(z_{k,\iota}, z_{k,\iota})^{-1/2}\| (1 - \|z_k\|^2) < r \quad (5.15)$$

where  $z_k = \bigoplus_\iota z_{k,\iota} \in \bigoplus_\iota V_\iota$ . If  $\|\xi_\iota\| = \lim_k \|z_{k,\iota}\| < 1$ , then this is obvious since

$$\limsup_{k \rightarrow \infty} \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(x_\iota)\|^2} = \frac{0}{1 - \|g_{-\xi_\iota}(x_\iota)\|^2} = 0$$

for any  $x_\iota \in D_\iota$ . If  $\|\xi_\iota\| = 1$  and if  $V_\iota$  is of finite rank, then Theorem 5.12 and (5.14) also implies such  $x_\iota \in D_\iota$  exists.

There remains the case of an *infinite-rank* Cartan factor  $V_\iota$  with  $\|\xi_\iota\| = 1$ . Such  $V_\iota$  can be realised as a closed subtriple of  $\mathcal{L}(H, H)$  for some Hilbert space  $H$ . Similar remark about the Bergmann operator  $B(a, a)$  as before allows us to assume  $V_\iota = \mathcal{L}(H, H)$  without loss of generality. Let  $t \in (0, 1)$ . For each  $k$ , define a bounded linear operator  $T_k : V_\iota \rightarrow V_\iota$  by

$$T_k(z) = (\mathbf{1} - tz_{k,\iota} z_{k,\iota}^*) z (\mathbf{1} - z_{k,\iota}^* z_{k,\iota}) \quad (z \in V_\iota = \mathcal{L}(H, H)).$$

Then by (1.2), we have  $T_k B(z_{k,\iota}, z_{k,\iota})^{-1/2}(z) = (\mathbf{1} - tz_{k,\iota} z_{k,\iota}^*)(\mathbf{1} - z_{k,\iota} z_{k,\iota}^*)^{-1/2} z (\mathbf{1} - z_{k,\iota}^* z_{k,\iota})^{1/2}$ . Since

$$\begin{aligned} & \|(\mathbf{1} - z_{k,\iota} z_{k,\iota}^*)^{-1/2} z (\mathbf{1} - z_{k,\iota}^* z_{k,\iota})^{1/2}\|^2 \\ &= \|(\mathbf{1} - z_{k,\iota} z_{k,\iota}^*)^{-1/2} z (\mathbf{1} - z_{k,\iota}^* z_{k,\iota}) z^* (\mathbf{1} - z_{k,\iota} z_{k,\iota}^*)^{-1/2}\| \\ &= \|B(z_{k,\iota}, z_{k,\iota})^{-1/2} (z (\mathbf{1} - z_{k,\iota}^* z_{k,\iota}) z^*)\| \leq \frac{\|z\|^2}{1 - \|z_{k,\iota}\|^2}, \end{aligned} \quad (5.16)$$

we infer  $\|T_k B(z_{k,\iota}, z_{k,\iota})^{-1/2}\| \leq 1/\sqrt{1 - \|z_{k,\iota}\|^2}$  and hence

$$\limsup_k \|B(x, x)^{-1/2} T_k B(z_{k,\iota}, z_{k,\iota})^{-1/2}\| (1 - \|z_k\|^2) \leq \lim_k \frac{\sqrt{1 - \|z_{k,\iota}\|^2}}{1 - \|x\|^2} = 0 \quad (5.17)$$

for all  $x \in D_\iota \subset V_\iota$ .

Let  $x_t = t\xi_\iota \in D_\iota$ . Then we have

$$\lim_k \|B(x_t, x_t)^{-1/2} B(x_t, z_{k,\iota}) - B(x_t, x_t)^{-1/2} T_k\| = \|B(x_t, x_t)^{-1/2} B(x_t, \xi_\iota) - B(x_t, x_t)^{-1/2} T\|$$

where  $T : V_\iota \rightarrow V_\iota$  is the operator  $T(z) = (\mathbf{1} - t\xi_\iota\xi_\iota^*)z(\mathbf{1} - \xi_\iota^*\xi_\iota)$  and for  $z \in V_\iota$ ,

$$\begin{aligned}
& \| (B(x_t, x_t)^{-1/2} B(x_t, \xi_\iota) - B(x_t, x_t)^{-1/2} T)(z) \| \\
&= \| (\mathbf{1} - t^2 \xi_\iota \xi_\iota^*)^{-1/2} (\mathbf{1} - t \xi_\iota \xi_\iota^*) z (\mathbf{1} - t \xi_\iota^* \xi_\iota - \mathbf{1} + \xi_\iota^* \xi_\iota) (\mathbf{1} - t^2 \xi_\iota^* \xi_\iota)^{-1/2} \| \\
&= \| (\mathbf{1} - t^2 \xi_\iota \xi_\iota^*)^{-1/2} (\mathbf{1} - t^2 \xi_\iota \xi_\iota^* + (t^2 - t) \xi_\iota \xi_\iota^*) z (\mathbf{1} - t) \xi_\iota^* \xi_\iota (\mathbf{1} - t^2 \xi_\iota^* \xi_\iota)^{-1/2} \| \\
&\leq \| (\mathbf{1} - t^2 \xi_\iota \xi_\iota^*)^{1/2} z (\mathbf{1} - t) \xi_\iota^* \xi_\iota (\mathbf{1} - t^2 \xi_\iota^* \xi_\iota)^{-1/2} \| \\
&\quad + \| (\mathbf{1} - t^2 \xi_\iota \xi_\iota^*)^{-1/2} (t^2 - t) \xi_\iota \xi_\iota^* z (\mathbf{1} - t) \xi_\iota^* \xi_\iota (\mathbf{1} - t^2 \xi_\iota^* \xi_\iota)^{-1/2} \| \\
&\leq \frac{(1-t)\|z\|}{\sqrt{1-t^2}\|\xi_\iota\|^2} + \frac{(t-t^2)(1-t)\|z\|}{1-t^2\|\xi_\iota\|^2}
\end{aligned}$$

where the last inequality follows from a computation similar to (5.16). This gives

$$\begin{aligned}
& \lim_k \| B(x_t, x_t)^{-1/2} B(x_t, z_{k,\iota}) - B(x_t, x_t)^{-1/2} T_k \| \\
&\leq \frac{1-t}{\sqrt{1-t^2}} + \frac{(t-t^2)(1-t)}{1-t^2} = \sqrt{\frac{1-t}{1+t}} + \frac{t(1-t)}{1+t}.
\end{aligned}$$

It follows from this inequality and (5.17) that

$$\limsup_k \| B(x_t, x_t)^{-1/2} B(x_t, z_{k,\iota}) B(z_{k,\iota}, z_{k,\iota})^{-1/2} \| (1 - \|z_k\|^2) \leq \sqrt{\frac{1-t}{1+t}} + \frac{t(1-t)}{1+t}$$

which can be made less than  $r$  by choosing  $t$  very close to 1. For such  $t$ , the element  $x_t \in D_\iota$  satisfies (5.15) and the proof is complete.  $\square$

## 6. ITERATION OF HOLOMORPHIC MAPS

The Denjoy-Wolff theorem holds for fixed-point free compact holomorphic self-maps on bounded symmetric domains of rank one, which asserts convergence of the iterates to a single boundary point [9, 15, 23]. The rank-one domains are the Hilbert balls of which the boundary points are exactly the *boundary components* (defined below) of the boundary. On the other hand, the Denjoy-Wolff theorem fails for the bidiscs. Instead of converging to a single boundary point, Hervé [16] has shown by intricate and lengthy arguments that the iterates accumulate in the closure of a single boundary component. This suggests that a natural generalisation of the Denjoy-Wolff theorem to other domains should be that the limit set of a fixed-point free compact holomorphic self-map  $f$  is contained in the closure of a single boundary component, where the limit set of  $f$  consists of the images of all subsequential limits  $\lim_k f^{n_k}$  of the iterates  $(f^n)$ , where  $f^n = \underbrace{f \circ \cdots \circ f}_{n\text{-times}}$ .

As in the case of the complex unit disc, the generalised Wolff theorem in Theorem 5.12 enables us to show in this section that for finite-rank bounded symmetric domains, all images of subsequential limits with weakly closed range are indeed contained in the closure of a single boundary component.

The concept of a boundary component of a convex domain  $U$  in a Banach space  $Z$  has been introduced and studied in [21, 22]. A subset  $C \subset \overline{U}$  is called a *boundary component* of the closure  $\overline{U}$  if the following conditions are satisfied:

- (i)  $C \neq \emptyset$ ;
- (ii) for each holomorphic map  $f : \mathbb{D} \rightarrow Z$  with  $f(\mathbb{D}) \subset \overline{U}$ , either  $f(\mathbb{D}) \subset C$  or  $f(\mathbb{D}) \subset \overline{U} \setminus C$ ;
- (iii)  $C$  is minimal with respect to (i) and (ii).

Two boundary components are either equal or disjoint. The interior  $U$  is the unique open boundary component of  $\overline{U}$ , all others are contained in the boundary  $\partial U$  [21]. By a slight abuse of language, we also called the latter boundary components of  $\partial U$ . For each  $a \in \overline{U}$ , we denote by  $K_a$  the boundary component containing  $a$ .

Given a holomorphic map  $h : U \rightarrow \overline{U}$ , the image  $h(U)$  is entirely contained in a single boundary component of  $\overline{U}$  (cf. [8, Lemma 3.3]).

For a bounded symmetric domain  $D$  realised as the open unit ball of a JB\*-triple  $V$ , the boundary component  $K_e$  of a tripotent  $e$  is given by  $K_e = e + V_0(e) \cap D$ , where  $V_0(e) = P_0(e)(V)$  is the Peirce 0-space. Moreover, if  $V$  is of finite rank, then each boundary component of  $\partial D$  is of this form (cf. [21, Proposition 4.3] and [22, Theorem 6.3]). Write  $D_e$  for the open unit ball  $V_0(e) \cap D$  in the JB\*-triple  $V_0(e)$ . Since  $P_0(e)$  is a contractive projection, we see that  $D_e = P_0(e)(D)$  and  $\overline{K}_e = e + \overline{D}_e = e + P_0(e)(\overline{D})$ . Also, the boundary  $\partial K_e$  of  $K_e$  equals  $e + \partial D_e$ .

Each tripotent  $c$  in  $V_0(e)$  is orthogonal to  $e$  and its Peirce 0-space in  $V_0(e)$  is the eigenspace

$$(V_0(e))_0(c) = \{v \in V_0(e) : (c \square c)(v) = 0\}.$$

**Lemma 6.1.** *In the above notation, we have  $(V_0(e))_0(c) = V_0(e + c)$ .*

*Proof.* Considering the joint Peirce decomposition induced by  $\{e, c\}$  and from (3.4), we have

$$V_0(e + c) = V_{00}(e, c) = V_0(e) \cap V_0(c) = (V_0(e))_0(c).$$

□

We see from Lemma 6.1 that each boundary component of  $\partial D_e$  is of the form  $c + (V_0(e))_0(c) \cap D_e = c + V_0(e + c) \cap D$  for some tripotent  $c \in V_0(e)$ . Hence each boundary component of  $\partial K_e$  is of the form  $e + c + V_0(e + c) \cap D = K_{e+c}$ , which is also a boundary component of  $\overline{D}$ .

Given a compact holomorphic self-map  $f$  on a bounded symmetric domain  $D$ , we call a holomorphic map  $h : D \rightarrow \overline{D}$  a *limit function* of the iterates  $(f^n)$  if there is a subsequence  $(f^{n_k})$  of  $(f^n)$  converging to  $h$  locally uniformly.

*Remark 6.2.* It has been shown in [9, Lemma 1] that every subsequence of  $(f^{n_k})$  has a subsequence converging locally uniformly to a holomorphic map  $h : D \rightarrow \overline{D}$ . It follows that if  $(f^n)$  has a unique limit function  $h$ , then  $(f^n)$  converges locally uniformly to  $h$ .

**Theorem 6.3.** *Let  $D$  be a bounded symmetric domain of finite rank  $p$  and let  $f : D \rightarrow D$  be a compact fixed-point free holomorphic map. Then there is a boundary point  $\xi \in \partial D$  of the form*

$$\xi = \sum_{j=1}^m \alpha_j e_j \quad (\alpha_j > 0, m \leq p)$$

for some orthogonal tripotents  $e_1, \dots, e_m \in \partial D$ , such that for each limit function  $h$  of  $(f^n)$  with weakly closed range, we have  $h(D) \subset \overline{K}_e$ , where  $K_e$  is the boundary component of  $e = e_1 + \dots + e_m$  in  $\partial D$ .

*Proof.* Let  $\xi = \sum_{j=1}^m \alpha_j e_j$  be the boundary point obtained in Theorem 5.12, where  $\alpha_j > 0$ ,  $m \leq p$  and  $e_1, \dots, e_m$  are orthogonal minimal tripotents. Let  $h$  be a limit function such that  $h(D)$  is weakly closed. Since  $f$  is a compact map, it follows from [18, Theorem 3.1] that  $h(D) \subset \partial D$  (cf. [8, Lemma 6.5]). By previous remarks,  $h(D)$  is contained in a boundary component  $K_u$  of  $\overline{D}$  for some tripotent  $u \in \partial D$ .

For  $n = 1, 2, \dots$ , pick  $y_n$  in the horoball  $S_0(\xi, n)$ . By  $f$ -invariance, we have  $h(y_n) \in S(\xi, n)$ , which, from Theorem 5.12, is of the form

$$h(y_n) = \sum_{j=1}^m \frac{\sigma_j n}{1 + \sigma_j n} e_j + B \left( \sum_{j=1}^m \sqrt{\frac{\sigma_j n}{1 + \sigma_j n}} e_j, \sum_{j=1}^m \sqrt{\frac{\sigma_j n}{1 + \sigma_j n}} e_j \right)^{1/2} (w_n)$$

for some  $w_n \in \overline{D}$ . Let  $(w_{n_k})$  be a subsequence of  $(w_n)$  weakly converging to  $w \in \overline{D}$ , say. Then the sequence  $(h(y_{n_k}))$  weakly converges to

$$\sum_{j=1}^m e_j + B \left( \sum_{j=1}^m e_j, \sum_{j=1}^m e_j \right)^{1/2} (w) = \sum_{j=1}^m e_j + P_0 \left( \sum_{j=1}^m e_j \right) (w) \in \overline{K}_e$$

where  $K_e$  is the boundary component in  $\partial D$  containing the tripotent  $e = e_1 + \dots + e_m$ . Since  $h(D)$  is weakly closed, we have  $\emptyset \neq h(D) \cap \overline{K}_e \subset K_u \cap \overline{K}_e$  and  $K_u$  meets either  $K_e$  or a boundary component of  $\partial K_e$ . By the remark following Lemma 6.1, the latter is also a boundary component of  $\overline{D}$ . It follows that either  $K_u = K_e$  or  $K_u$  is a boundary component of  $\partial K_e$ , that is,  $K_u \subset \overline{K}_e$  which gives  $h(D) \subset \overline{K}_e$ . □

*Remark 6.4.* The above result generalises the Denjoy-Wolff theorem for Hilbert balls. Indeed, for a compact fixed-point free holomorphic self-map  $f$  on a Hilbert ball  $D$ , we have  $\xi = e_1$  and  $K_{e_1} = \{\xi\}$  in the above theorem. Each subsequential limit  $h$  of  $(f^n)$  is constant [9, p.1775] and hence  $h(D) = \{\xi\}$ . This implies locally uniform convergence of  $(f^n)$  to the constant map taking value  $\xi$ , by Remark 6.2.

**Example 6.5.** Although it is known that the Denjoy-Wolff Theorem fails for a non-compact holomorphic self-map on a Hilbert ball [27], we see in the following example that compactness is not a necessary condition for a Denjoy-Wolff type theorem (see also [10]). This example also shows that the image  $h(D)$  of a limit function  $h$  can be a singleton or a whole boundary component.

Let  $D$  be a finite-rank bounded symmetric domain of rank  $p$ . Pick any nonzero  $a \in D$ , with spectral decomposition

$$a = \alpha_1 e_1 + \dots + \alpha_p e_p \quad (\|a\| = \alpha_1 \geq \dots \geq \alpha_p \geq 0).$$

Let  $g_a : D \rightarrow D$  be the Möbius transformation induced by  $a$ , which is not a compact map if  $D$  is infinite dimensional. Let  $x = \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_p e_p$ , where  $\beta_1, \beta_2, \dots, \beta_p \in \mathbb{D}$  so that  $x \in D$ . By orthogonality, we have

$$x \square a = (\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_p e_p) \square (\alpha_1 e_1 + \dots + \alpha_p e_p) = \beta_1 \alpha_1 e_1 \square e_1 + \dots + \beta_p \alpha_p e_p \square e_p$$

and

$$(x \square a)^n(x) = \beta_1^{n+1} \alpha_1^n e_1 + \dots + \beta_p^{n+1} \alpha_p^n e_p \quad (n = 1, 2, \dots).$$

It follows that

$$\begin{aligned}
g_a(x) &= a + B(a, a)^{1/2}(\mathbf{1} + x \square a)^{-1}(x) \\
&= a + B(a, a)^{1/2}(\mathbf{1} - x \square a + (x \square a)^2 - (x \square a)^3 + \cdots)(x) \\
&= a + B(a, a)^{1/2}(\beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_p e_p - (\beta_1^2 \alpha_1 e_1 + \cdots + \beta_p^2 \alpha_p e_p) + \cdots) \\
&= a + B(a, a)^{1/2}[(1 - \beta_1 \alpha_1 + \beta_1^2 \alpha_1^2 + \cdots)\beta_1 e_1 + \cdots + (1 - \beta_p \alpha_p + \beta_p^2 \alpha_p^2 + \cdots)\beta_p e_p] \\
&= a + B(a, a)^{1/2} \left( \frac{\beta_1 e_1}{1 + \beta_1 \alpha_1} + \cdots + \frac{\beta_p e_p}{1 + \beta_p \alpha_p} \right) \\
&= \alpha_1 e_1 + \cdots + \alpha_p e_p + \frac{(1 - \alpha_1^2)\beta_1 e_1}{1 + \beta_1 \alpha_1} + \cdots + \frac{(1 - \alpha_p^2)\beta_p e_p}{1 + \beta_p \alpha_p} \\
&= \frac{\alpha_1 + \beta_1}{1 + \alpha_1 \beta_1} e_1 + \cdots + \frac{\alpha_p + \beta_p}{1 + \alpha_p \beta_p} e_p \\
&= \mathfrak{g}_{\alpha_1}(\beta_1) e_1 + \cdots + \mathfrak{g}_{\alpha_p}(\beta_p) e_p
\end{aligned}$$

where  $\mathfrak{g}_{\alpha_j}$  is the Möbius transformation on the complex disc  $\mathbb{D}$ , induced by  $\alpha_j$  for  $j = 1, \dots, p$ . If  $\alpha_j = 0$ , then  $\mathfrak{g}_{\alpha_j}$  is the identity map. If  $\alpha_j > 0$ , then the iterates  $(\mathfrak{g}_{\alpha_j}^n)$  converge locally uniformly to the constant map with value  $\alpha_j/|\alpha_j| = 1$ . Hence the iterates

$$g_a^n(x) = \mathfrak{g}_{\alpha_1}^n(\beta_1) e_1 + \cdots + \mathfrak{g}_{\alpha_p}^n(\beta_p) e_p \quad (n = 2, 3, \dots)$$

converge to

$$e_1 + \gamma_2 e_2 + \cdots + \gamma_p e_p, \quad \gamma_j = \begin{cases} 1 & (\alpha_j > 0) \\ \beta_j & (\alpha_j = 0) \end{cases} \quad (j = 2, \dots, p).$$

In particular, if  $\alpha_j > 0$  for all  $j$ , then by Remark 6.2, the iterates  $(g_a^n)$  converge locally uniformly to a constant map with value  $\xi = e_1 + \cdots + e_p$  which is a maximal tripotent in  $\partial D$ . On the other hand, if  $J = \{j : \alpha_j > 0\}$  is a proper subset of  $\{1, \dots, p\}$ , then

$$\lim_n g_a^n(x) = \sum_{j \in J} e_j + \sum_{j \notin J} \beta_j e_j \in e + D_e$$

where  $e = \sum_{j \in J} e_j$  is a tripotent in  $\partial D$  and  $D_e = V_0(e) \cap D$ . It follows that, in this case, the image of every limit function  $h$  of  $(g_a^n)$  is the whole boundary component  $e + D_e$  since for any  $e + z \in e + D_e$  with  $z \in D_e$  and spectral decomposition  $z = \sum_{j \notin J} \beta_j u_j$ , we have  $h(\sum_{j \in J} \alpha_j e_j + \sum_{j \notin J} \beta_j u_j) = e + \sum_{j \notin J} \beta_j u_j$ .

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